HOMEWORK #9

DUE FRIDAY APRIL 16TH

(1) Determine whether the following series converges and whether it converges absolutely.
 (a) $\sum_{n=1}^{\infty} \frac{(1-i)^n}{n!}$
 (b) $\sum_{n=1}^{\infty} (1/3 + 2/3i)^n$

(2) Use the ratio test to find the radius of convergence of the following power series. *Justify* your answer!
 (a) $\sum_{n=1}^{\infty} \frac{z^n}{n^3}$
 (b) $\sum_{n=1}^{\infty} z^{2n}$

(3) Suppose that $\{a_n\}$ is a sequence of complex numbers converging to L. Further suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous. Show that the sequence $\{f(a_n)\}$ convergens to $f(L)$.

(4) Show that every complex number z_0 such that $|z_0| = 1$ can be written as e^{ir} for some real number $r \in \mathbb{R}$.

(5) Assume that $e^z \cdot e^w = e^{z+w}$ for all complex numbers z and w (you need not prove it, but it is true). Prove that the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!}$ is *NOT* injective. What does this say about finding a natural log function on $\mathbb{C}$? Furthermore, show that the image of f is equal to $\mathbb{C} \setminus 0$.

(6) Suppose that $\{a_n\}$ is a convergent sequence of complex numbers. Prove that the sequence is bounded. In other words, show that there exists a real number $M > 0$ such that $|a_n| \leq M$ for every n.

(7) A sequence of complex numbers $\{a_n\}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists a $N > 0$ such that if $m, n > N$, then $|a_m - a_n| < \varepsilon$. Show that a sequence of complex numbers is convergent if and only if it is Cauchy.

(8) Use the Cauchy Riemann equations to show that the following functions are not differentiable.
 (a) $f(z) = f(x + iy) = 2y - ix$.
 (b) $g(z) = g(x + iy) = z^2 - ix$.

(9) You will be given a function $u(z) = u(x+iy)$ below. Find a function $v(z) = v(x+iy)$ such that $f(z) = u(z) + iv(z)$ satisfies the Cauchy Riemann equations for all $z \in \mathbb{C}$. Alternately, explain why it can't be done.
 (a) $u(x + iy) = x + y$.
 (b) $u(x + iy) = x^2$.
 (c) $u(x + iy) = e^x$.

(10) Prove that a non-constant, complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ cannot only output real numbers.