HOMEWORK #7

DUE FRIDAY MARCH 19TH

(1) Fix $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ to be a basis. Find the eigvalues and describe the eigenvectors of the following linear transformations.
  (a) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{u}) = \mathbf{v}$ and $T(\mathbf{v}) = 2\mathbf{v}$.
  (b) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.
  (c) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$.
  (d) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$.

(2) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and that $S : \mathbb{R}^m \to \mathbb{R}^l$ is another linear transformation. Prove that $S \circ T$ is also a linear transformation.

(3) Fix $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ to be a basis. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and that $S : \mathbb{R}^2$ is represented by the matrix $\begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$. Write down the matrix representation for $T \circ S$. Also write down the matrix representation for $S \circ T$. (What you have just done is called matrix multiplication).

(4) Suppose that $U : \mathbb{R}^2$ is a linear transformation. Further suppose that for every other linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, we know that $T \circ U = U \circ T$. What can you say about U? Prove it!

(5) Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ is a basis. Further suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ is another basis. Explain how you would construct a matrix which represents (with respect to the first basis $\mathbf{u}, \mathbf{v}$) the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the property that $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y}$. Conversely, explain how you would construct a matrix which represents (with respect to the first basis $\mathbf{u}, \mathbf{v}$) the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the property that $S(\mathbf{x}) = \mathbf{u}$ and $S(\mathbf{y}) = \mathbf{v}$. What is the matrix representation for $S \circ T$? What is the matrix representation for $T \circ S$?

(6) Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ is a basis. Further suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ is another basis and suppose S and T are as in problem (3). Finally let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be another linear transformation and suppose that M is represented by a matrix A (with respect to the basis $\mathbf{x}, \mathbf{y}$). Show that the matrix representation of $S \circ M \circ T$ (with respect to $\{\mathbf{x}, \mathbf{y}\}$) is exactly the same as the matrix representation of M with respect to $\{\mathbf{u}, \mathbf{v}\}$.

(7) Fix an orthnormal basis $\mathbf{u}, \mathbf{v}$ and suppose that $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations (which you may want to represent / "think of" as matrices). Prove that $\det(A \circ B) = \det(A) \cdot \det(B)$. Given the determinant's geometric interpretation as a way to measure change in area, describe in words why this is formula is you would expect.