HOMEWORK #6

DUE MONDAY MARCH 8TH

(1) Write down matrix representations of the following linear transformations. Also explain as well as you can what this linear transformation does geometrically. Fix an orthonormal basis $\mathbf{u}, \mathbf{v}$ for $\mathbb{R}^2$ and an orthonormal basis $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for $\mathbb{R}^3$.
   (a) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{u}) = \mathbf{v}$ and $T(\mathbf{v}) = 2\mathbf{v}$.
   (b) The map $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined in the following way. $T(\mathbf{u}) = 1$ and $T(\mathbf{v}) = -1$.
   (c) The map $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined in the following way. $T(\mathbf{u}) = \mathbf{0}$ and $T(\mathbf{v}) = \mathbf{x}$.
   (d) The map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{x}) = \mathbf{v}$ and $T(\mathbf{y}) = \mathbf{u}+2\mathbf{v}$ and $T(\mathbf{z}) = 2\mathbf{u} - \mathbf{v}$.

(2) Fix a basis $\mathbf{u}, \mathbf{v}$ for $\mathbb{R}^2$ and a basis $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for $\mathbb{R}^3$. In the following problem you will be given a linear tranformation T represented by a matrix and you will also be given a vector $\mathbf{w}$. Compute $T(\mathbf{w})$ and write your answer as both a column vector and as a linear combination of the appropriate basis vectors.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$ (here K is a constant real number), and $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$.
   (c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ is represented by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, and $\mathbf{w} = \mathbf{u}$.
   (d) $T : \mathbb{R}^3 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{w} = \mathbf{x} + \mathbf{y} + \mathbf{z}$.
   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(3) Suppose that T is a linear transformation that is *not* injective. Prove that there exists a non-zero element $\mathbf{u}$ in the domain of T such that $T(\mathbf{u}) = \mathbf{0}$.

(4) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation $n = 2$ or 3, $m = 2$ or 3. Suppose that $\mathbf{x}_1, \dots \mathbf{x}_n$ is a basis for the domain of T. Show that T is injective if and only if $T(\mathbf{x}_1), \dots T(\mathbf{x}_n)$ is a linearly independent set in $\mathbb{R}^m$.

(5) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation $n = 2$ or 3, $m = 2$ or 3. Suppose that $\mathbf{x}_1, \dots \mathbf{x}_n$ is a basis for the domain of T. Show that T is surjective if and only if $T(\mathbf{x}_1), \dots T(\mathbf{x}_n)$ is a spanning set in $\mathbb{R}^m$.

(6) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation represented by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that T is injective if and only if T is surjective (see the worksheet from last semester, due November 18th, for the answer to this). Prove also that T is injective if and only if $ad - bc \neq 0$.