HOMEWORK #4

DUE WEDNESDAY FEBRUARY 3RD

(1) Give example of the following types of sequences.
  (a) An unbounded sequence that has a convergent subsequence.
  (b) A bounded sequence that has two different convergent subsequences which converge to distinct values.

(2) Consider the following sequences.
  (a) Suppose that a_0 and a_1 are distinct real numbers. For each $n > 1$ define
$$a_n = (a_{n-1} + a_{n-2})/2$$
  (so a_n depends on the previous values). Show that $\{a_n\}$ is a Cauchy sequence.
  (b) Consider the sequence $\{b_n\}$ defined by the rule
$$b_n = \sum_{i=0}^{n} \frac{1}{2^i}$$
  Prove it converges by showing it is a Cauchy sequence.

(3) Let us take the following statement as an axiom:
  "Every Cauchy sequence is convergent."
But do not assume that every set that is bounded above has a least upper bound. Prove then (using the fact about Cauchy sequences) that every set that is bounded above has a least upper bound.
Hint: The Bolzano-Weierstrass theorem (and the Lemma used to prove it) may help. Go through the theorems in the chapter and make sure you aren't assuming the least upper bound axiom as well.

(4) Use the ratio test, the comparison test, or the ingegral test to decide whether the following series are convergent.
  (a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
  (b) $\sum_{n=1}^{\infty} \frac{n^3}{n!}$
  (c) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$
  (d) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$
  (e) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
  (f) $\sum_{n=1}^{\infty} \frac{(4^n)n!}{n^n}$
  (g) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

(5) Suppose that the partial sums of the sequence $\{a_n\}$ are bounded. Further suppose that $\{b_n\}$ is a decreasing sequence that converges to 0. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

(6) Suppose that $a_n \geq 0$ for all n and also that $\sum_{n=1}^{\infty} a_n$ diverges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.

(7) Suppose that $\{a_n\}$ is a non-increasing sequence (in other words $a_k \geq a_{k+1}$ for all k) of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Show that the series
$$\sum_{n=1}^{\infty} a_n$$
converges if and only if the series
$$\sum_{n=1}^{\infty} 2^n a_{(2^n)}$$
converges.