HOMEWORK #3
MATH 186–1
WINTER 2010

DUE WEDNESDAY JANUARY 27TH

(1) Suppose that f'' exists.
   (a) Show that
   $$f''(a) = \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$
   Hint: Use the Taylor polynomial $P_{2,a}(x)$ with $x = a + h$ and with $x = a - h$.
   (b) Let
   $$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$
   Show that f'' does not exist (at $x = 0$) but that $\lim_{h \to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}$ exists.

(2) Suppose that a_i and b_i are coefficients in the Taylor polynomials of f and g respective (at 0). Thus $a_i = f^{(i)}(0)/(i!)$ and $b_i = g^{(i)}(0)/(i!)$. Find the Taylor coefficients c_i of the following functions (at 0) in terms of the a_i and b_i.
   (i) $f + g$
   (ii) $f - g$
   (iii) f'
   (iv) $f \cdot g$ (this one may take some thought)
   (v) $h(x) = \int_0^x f(t)dt$

(3) What can be said about the sequence $\{a_n\}$ if it converges and each a_n is an integer?

(4) Suppose that $\{a_n\}$ is a convergent sequence. Prove that the set of all numbers a_n is bounded. Further show that at least one of the following are true.
   (i) The set of all numbers a_n has a maximum value.
   (ii) The set of all numbers a_n has a minimum value.

(5) Suppose that f is continuous and that the sequence
   $$x, f(x), f(f(x)), f(f(f(x))), \ldots$$
   converges to c. Prove that c is a "fixed point" for f. In other words, show that $f(c) = c$.

(6) Let $\{x_n\}$ be a bounded sequence and set $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Show that $\{y_n\}$ converges.