HOMEWORK #11
MATH 185–4
FALL 2009

DUE WEDNESDAY, DECEMBER 9TH

(1) Decide which of the following functions are integrable on $[0, 2]$ and then compute the integral if possible (without using the fundamental theorem of calculus). Justify your answers.
  (a) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x - 2, & 1 < x \le 2 \end{cases}$
  (b) $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ \frac{1}{x}, & 1 < x \le 2 \end{cases}$
  (c) $f(x) = \begin{cases} 0, & x \ne 1 \\ 38, & x = 1 \end{cases}$
  (d) $f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some integer } n > 0 \\ 0, & \text{otherwise} \end{cases}$
  (e) $f(x) = \begin{cases} 0, & x \text{ rational} \\ x, & x \text{ irrational} \end{cases}$

(2) Suppose that f is integrable on $[a, b]$ and that c is a real number. Prove that
$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx.$$
Hint, every partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$ induces a partition $P' = \{t_0 + c, \ldots, t_n + c\}$ of $[a + c, b + c]$.

(3) Suppose that f is integrable on $[a, b]$ and that $[c, d] \subseteq [a, b]$. Prove that f is integrable on $[c, d]$.

(4) First a definition.

Definition. A function s with domain $[a, b]$ is called a *step function* if there is a partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$ such that s is constant on each (t_{i-1}, t_i).

Now for the problem.
  (a) Prove that a sum of step functions is a step function. (We won't use it, this is just a warm-up).
  (b) Prove that if f is integrable on $[a, b]$, then for any $\varepsilon > 0$ there exists a step function s such that $s(x) \le f(x)$ for all x and such that $\left(\int_a^b f(x)dx\right) - \left(\int_a^b s(x)d(x)\right) < \varepsilon$.
  (c) Suppose for all $\varepsilon > 0$ there are step functions s_1, s_2 with $s_1(x) \le f(x)$ for all x and $s_2(x) \ge f(x)$ for all x such that
$$\int_a^b s_2(x)dx - \int_a^b s_1(x)dx < \varepsilon.$$
  Prove then that f is integrable.

(5) Suppose that f is integrable on $[a, b]$ and that $c > 0$ is a real number. Prove that
$$c\int_a^b f(cx)dx = \int_{ca}^{cb} f(x)dx.$$

(6) In section 2–9 in Kaplan and Lewis, do problems #1, 2, 8.