MATH 185-4, EXAM #1
SOLUTIONS

1. Definitions and short answers.
(a) Give a precise definition of what it means for a function f with domain $[a, b]$ to be *continuous on* $[a, b]$. (5 points)

A function f with domain $[a, b]$ is said to be continuous on $[a, b]$ if all of the following conditions hold:

- for all $c \in (a, b)$, $\lim_{x \to c} f(x) = f(c)$.
- $\lim_{x \to a^+} f(x) = f(a)$.
- $\lim_{x \to b^-} f(x) = f(b)$.

(b) Give a precise definition of the following term: *surjective function* (5 points)

A function f is called surjective if for every $c \in \mathbb{R}$, there exists an element d in the domain of f such that $f(d) = c$.

(c) Give a precise definition of the following term: *linearly independent set of vectors in the plane* (5 points)

A pair of vector $\mathbf{u}, \mathbf{v}$ in the plane are said to be linearly independent if whenever one has an equation $a\mathbf{u} + b\mathbf{v} = 0$ for real numbers a and b, then both $a = 0$ and $b = 0$.

(d) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors. Explain using words how to define $\mathbf{u} + \mathbf{v}$ (you may use a picture to help illustrate). (5 points)

Fix a point A and realize the vector $\mathbf{u}$ as $\vec{AB}$ for some point B. Realize the vector $\mathbf{v}$ as $\vec{BC}$ where the point B is as before. Define $\mathbf{u} + \mathbf{v}$ to be $\vec{AC}$.

2 Use ε's and δ's in part (a)
(a) (20 points) Show that:

$$\lim_{x \to 0}(x^3 - 2x) = 0$$

Fix $\varepsilon > 0$. Choose $\delta = \min(1, \varepsilon/3)$. Suppose that x satisfies $0 < |x - 0| < \delta$. Write $f(x) = x^3 - 2x$, $L = 0$ and $a = 0$.

Therefore, we see that $|x| < \delta \leq 1$ so $|x| < 1$. But then $|x^2 - 2| \leq |x^2| + |-2| = |x|^2 + 2 < 1^2 + 2 = 3$.

On the other hand $|x| < \varepsilon/3$ so that $3|x| < \varepsilon$. But then

$$|f(x) - 0| = |x^3 - 2x - 0| = |x^2 - 2| \cdot |x| < 3|x| < \varepsilon$$

which completes the proof.

2(b) Define

$$g(x) = \begin{cases} x^3 - 2x, & x > 0 \\ x, & x < 0 \end{cases}$$

Find a $\delta > 0$ such that whenever $0 < |x| < \delta$, then $|g(x) - 0| < 0.5$ (ie, what δ works for $\varepsilon = 0.5$)? Justify your answer (10 points)

Set $\delta = 0.5/3 = \frac{1}{6}$. Suppose that x satisfies $0 < |x| < \delta$. We have two cases.

- $x > 0$ In this case $g(x) = f(x)$ where $f(x)$ is as in the proof of part (a). But then we just showed that if $|x| < \frac{1}{6} = \varepsilon/3$ (for $\varepsilon = 0.5$), then $|g(x) - 0| = |f(x)| < \epsilon = 0.5$. As desired.
- $x < 0$ In this case $g(x) = x$. But then if $|x| < \delta$, then $x \in (-1/6, 0)$. We now want to show that $|x| < 0.5$ but that is obvious since $1/6 < 0.5$.

3. Suppose that f is a continuous function on $\mathbb{R}$. Suppose also that $f(0) = -2$ and that $f(1) = 7$. Consider the following set:

$$S := \{x \in [0,1] | f(x) = 0\}$$

(a) Explain why S is not empty. (8 points)

By the intermediate value theorem, there exists some $c \in [0,1]$ such that $f(c) = 0$. Thus $c \in S$ so $S \neq \emptyset$.

(b) Set $\alpha = \sup(S)$ (ie, α is the least supper bound for S). Explain why $\alpha \in [0,1]$. (5 points)

First note that 1 is an upper bound for S and so $\alpha \leq 1$. On the other hand, the c from part (a) is in S and $c \in S \subseteq (0,1)$ which means that $c > 0$. But then $\alpha \geq c > 0$ so $\alpha > 0$. Combining these two facts we see that $\alpha \in (0,1]$ (an even better statement than the problem asked for).

(c) Show that $f(\alpha) = 0$. (12 points)

Hint: Suppose that $f(\alpha) \neq 0$, and then use the following lemma which you may cite without proof

Lemma: If f is continuous at α and $f(\alpha) \neq 0$, then there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $x \in (\alpha - \delta, \alpha + \delta)$.

Use the lemma to contradict the choice of α.

Suppose $\alpha \notin S$ or in other words suppose that $f(\alpha) \neq 0$. By the Lemma there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $x \in (\alpha - \delta, \alpha + \delta)$. In particular, no element of $(\alpha - \delta, \alpha + \delta)$ is in S (ie $(\alpha - \delta, \alpha + \delta) \cap S = \emptyset$).

We showed in class that if α is an upper bound for a set S, then for every $\varepsilon > 0$, we have that $(\alpha - \varepsilon, \alpha] \cap S \neq \emptyset$. But set $\varepsilon = \delta$ and then $\emptyset \neq (\alpha - \delta, \alpha] \cap S \subseteq (\alpha - \delta, \alpha + \delta) \cap S$. In particular $(\alpha - \delta, \alpha + \delta) \cap S \neq \emptyset$ but that contracts what we wrote above.

4. (25 points) Suppose that g is a function that is continuous of a and that f is a function that is continuous at $g(a)$. Prove that $f \circ g$ is continuous at a using δ's and ε's.

I'll refer you to chapter 6 of the book for this.

(Extra Credit) (10 points) We say that a set $D \subseteq \mathbb{R}$ is *closed* if $\mathbb{R} \setminus D = \{x \in \mathbb{R} | x \notin D\}$ is open.

Suppose that f is continuous with domain $\mathbb{R}$. Prove that for every closed set $D \subseteq \mathbb{R}$, $f^{-1}(D)$ is also closed.

To show that $f^{-1}(D)$ is closed we need to show that $\mathbb{R} \setminus f^{-1}(D)$ is open. We will show this by showing that $\mathbb{R} \setminus f^{-1}(D) = f^{-1}(\mathbb{R} \setminus D)$ (which is open because f is continuous and $\mathbb{R} \setminus D$ is open by assumption).

First we will show that $\mathbb{R} \setminus f^{-1}(D) \subseteq f^{-1}(\mathbb{R} \setminus D)$. So choose $x \in \mathbb{R} \setminus f^{-1}(D)$. Thus $f(x) \notin D$. Therefore $f(x) \in \mathbb{R} \setminus D$ which implies that $x \in f^{-1}(\mathbb{R} \setminus D)$ as desired.

Now we show that $\mathbb{R} \setminus f^{-1}(D) \supseteq f^{-1}(\mathbb{R} \setminus D)$. So choose $x \in f^{-1}(\mathbb{R} \setminus D)$. Then $f(x) \in \mathbb{R} \setminus D$. In particular, $f(x) \notin D$. Therefore x can't be in $f^{-1}(D)$. But then $x \in \mathbb{R} \setminus f^{-1}(D)$ again as desired.

Now we know that both $\mathbb{R} \setminus f^{-1}(D) \supseteq f^{-1}(\mathbb{R} \setminus D)$ and $\mathbb{R} \setminus f^{-1}(D) \subseteq f^{-1}(\mathbb{R} \setminus D)$. Therefore

$$\mathbb{R} \setminus f^{-1}(D) = f^{-1}(\mathbb{R} \setminus D).$$

In particular $\mathbb{R} \setminus f^{-1}(D)$ is open and so $f^{-1}(D)$ is closed.