# EXTRA PROBLEMS #7

DUE: FRI NOVEMBER 9TH

We'll be doing some exercises related to vector functions. See sections 3-10 through 3-12 in the text for additional discussion.

A *vector function* is a function $\mathbf{F}$ that takes a real number in as input, and outputs a vector in the plane.

**Exercise 0.1.** Show that every vector function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ can be described by a rule

$$\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions of a single variable.

*Proof.* For each vector $\mathbf{u} \in \mathbb{R}^2$, we know that there exist unique scalars $r_{\mathbf{u}}$ and $s_{\mathbf{u}}$ such that $\mathbf{u} = r_{\mathbf{u}}\mathbf{i} + s_{\mathbf{u}}\mathbf{j}$. The $r_{\mathbf{u}}$ and $s_{\mathbf{u}}$ are actually functions of $\mathbf{u}$ that take in vectors, and output scalars (the uniqueness of the $r$ and $s$ make this really a function). Thus we define $f(t) = r_{\mathbf{F}(t)}$ and $g(t) = s_{\mathbf{F}(t)}$. $\square$

The following definition can be found on pages 187 and 188 of the text. We say that

$$\lim_{t \to c} \mathbf{F}(t) = \mathbf{v}$$

if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $t \in (c-\delta, c+\delta)$, $t \neq c$, then $|\mathbf{F}(t) - \mathbf{v}| < \epsilon$.

This second condition can be rephrased as requiring that $\mathbf{F}(t)$ is inside the circle of radius $\epsilon$ centered at $\mathbf{v}$.

We can also define what it means for a vector function to be continuous. A vector function $\mathbf{F}(t)$ is said to be *continuous at* $t = c$ if

$$\lim_{t \to c} = \mathbf{F}(c).$$

If $\mathbf{F}$ is continuous at every point in its domain, we simply say that $\mathbf{F}$ is *continuous*.

**Definition 0.2.** We say that a subset $U \subseteq \mathbb{R}^2$ is *open* if for every point $P \in U$, there exists an $\epsilon > 0$ such that the interior of the disc of radius $\epsilon$ centered at $P$ is completely contained in $U$.

In other words, if $P = (p_1, p_2)$, then all points $\mathbf{v} = (x_1, x_2)$ which satisfy the inequality

$$|\mathbf{v} - P| = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2} < \epsilon,$$

are contained inside $U$.

**Exercise 0.3.** Prove that $U \subset \mathbb{R}^2$ is open if and only if for every point $P \in U$, there exists a $e > 0$ such that the interior of the square with side length $2e$ and with central point $P$ is completely contained in $U$.

*Proof.* We have to prove both "dicrections". First suppose that $U$ is open and choose a point $P \in U$. We need to find a square around $P$ that is completely contained in $U$. Since $U$ is open, there exists a $\epsilon > 0$ such that the interior of the disc of radius $\epsilon$ centered at $P$

is contained in $U$. We just need to find an $e$ so that the square of side-length $2e$ (centered at $P$) is contained inside that disk (which is inside $U$). A quick diagram shows that letting $e = \frac{\epsilon}{\sqrt{2}}$ works. One can also easily check this algebraically.

To prove the other direction, roughly speaking, we need to show that if every point has a square around it, it also has a disk around it. Fix a $P \in U$. Let $e > 0$ be such that the interior of the square centered at $P$ with side-length $2e$ is contained inside $U$. Choose $\epsilon = e$ and note that the disk of radius $\epsilon$ is completely contained inside the square (one can again either do this algebraically, or with a picture). □

See extra problems #4 for the definition of an open subset $U$ of $\mathbb{R}$. Also see extra problems #4 for the definition of $f^{-1}(U)$ where $f$ is a function and $U$ is a subset of the codomain.

**Exercise 0.4.** Prove that a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is continuous if and only if for every open subset $U \subseteq \mathbb{R}^2$, the subset of $\mathbb{R}$, $\mathbf{F}^{-1}(U)$, is also open.

*Hint:* Try something similar to the proofs of 2.2 and 2.3 in extra problems #4.

*Proof.* Again we have to prove two directions. First suppose that $\mathbf{F}$ is continuous. Let $U$ be an open subset of $\mathbb{R}^2$, and fix a point $x_0 \in \mathbf{F}^{-1}(U)$. Note we need to show that $\mathbf{F}^{-1}(U)$ is open. In other words, $P = \mathbf{F}(x_0) \in U$. Since $U$ is open, there exists an $\epsilon > 0$ such that the interior of the disk of radius $\epsilon$ centered at $P$ (which we will denote by $D_\epsilon(P)$ is contained in $U$. Since $\mathbf{F}$ is continuous, there exists a $\delta$ such that $\mathbf{F}(x) \in D_\epsilon(P)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Because $D_\epsilon(P) \subseteq U$, we see that $(x_0 - \delta, x_0 + \delta) \subseteq \mathbf{F}^{-1}(U)$. But, since $x_0 \in \mathbf{F}^{-1}(U)$ was arbitrary, this proves that $\mathbf{F}^{-1}(U)$ was open, as desired.

Conversely, suppose that for every open $U \subseteq \mathbb{R}^2$, we know that $\mathbf{F}^{-1}(U)$ is an open subset of $\mathbb{R}$. First we claim that any interior of any disk in $\mathbb{R}^2$ is itself an open set. This is very easy to convince yourself of graphically, and it can be verified algebraically as well because one can consider every point $P$ on the disk as on a radial line through the center $C$. Then it is easy to determine the proper radius to create a smaller disk around $P$ inside the larger disk $C$.

Now back to the proof. Given a $x_0 \in \mathbb{R}$ and a corresponding point $P = \mathbf{F}(x_0) \in \mathbb{R}^2$, and given an $\epsilon > 0$, define $U$ to be the disk of radius $\epsilon$ centered at $P$. By hypothesis, $\mathbf{F}^{-1}(U)$ an is open subset of $\mathbb{R}$. Also note that $x_0 \in \mathbf{F}^{-1}(U)$ since $\mathbf{F}(x_0) = P \in U$. So there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset \mathbf{F}^{-1}(U)$. But then, for all $x \in (x_0 - \delta, x_0 + \delta)$, we have $x \in \mathbf{F}^{-1}(U)$, which means $\mathbf{F}(x) \in U =$ "The interior of the disk of radius epsilon centered around$\mathbf{F}(x_0)$''. We have thus proven that $\mathbf{F}$ is continuous since $x_0$ was arbitrary. □

Given a vector function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$, we say that $\mathbf{F}$ is *differentiable at* $t_0 \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{1}{h} \left( \mathbf{F}(x_0 + h) - \mathbf{F}(x_0) \right)$$

exists. If the limit exists, we define $\mathbf{F}'(t_0)$ to be the value of that limit, and we say $\mathbf{F}'(t_0)$ is the *derivative of* $\mathbf{F}(t)$ *at* $t_0$. See section 3-11 in the text.

**Exercise 0.5.** Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is differentiable at every point, prove that $\mathbf{F}$ is also continuous at every point.

*Proof.* Fix $x_0 \in \mathbb{R}$. We know that

$$\lim_{x \to x_0} \left( \frac{1}{x - x_0} \left( \mathbf{F}(x) - \mathbf{F}(x_0) \right) \right) = \lim_{h \to 0} \left( \frac{1}{h} (\mathbf{F}(x_0 + h) - \mathbf{F}(x_0)) \right) = \mathbf{F}'(x_0) = \lim_{x \to x_0} \mathbf{F}'(x_0)$$

Thus, by multiplication by $x - x_0$, we see that

$$\lim_{x\to x_0}(\mathbf{F}(x)-\mathbf{F}(x_0)) = \lim_{x\to x_0}\left(\frac{x-x_0}{x-x_0}(\mathbf{F}(x)-\mathbf{F}(x_0))\right) = \lim_{x\to x_0}(x-x_0)\mathbf{F}'(x_0) = 0$$

Therefore, by adding $\mathbf{F}(x_0)$ to both sides we see that

$$\lim_{x\to x_0}\mathbf{F}(x) = \mathbf{F}(x_0)$$

as desired. □

**Exercise 0.6.** Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is a vector function that is differentiable at $t_0$. Suppose further that $g : \mathbb{R} \to \mathbb{R}$ is a function of a single variable that is also differentiable at $t_0$.

Consider a new function $\mathbf{H}(t) = g(t)\mathbf{F}(t)$ (here we are viewing $g(t)$ as a scalar). Prove that $\mathbf{H}'(t) = g'(t)\mathbf{F}(t) + g(t)\mathbf{F}'(t)$.

*Proof.* If it exists,

$$\mathbf{H}'(t) = \lim_{h\to 0}\left(\frac{1}{h}(\mathbf{H}(t+h)-\mathbf{H}(t))\right) = \lim_{h\to 0}\left(\frac{1}{h}(g(t+h)\mathbf{F}(t+h)-g(t)\mathbf{F}(t))\right)$$

Using the same trick as in the single variable case, we note that we can write this as

$$\lim_{h\to 0}\left(\frac{1}{h}\left(g(t+h)\mathbf{F}(t+h)-g(t+h)\mathbf{F}(t)+g(t+h)\mathbf{F}(t)-g(t)\mathbf{F}(t)\right)\right).$$

After algebraic manipulation, this becomes

$$\lim_{h\to 0}\left(\frac{1}{h}\left(g(t+h)\mathbf{F}(t+h)-g(t+h)\mathbf{F}(t)\right)\right)+\lim_{h\to 0}\left(\frac{1}{h}\left(g(t+h)\mathbf{F}(t)-g(t)\mathbf{F}(t)\right)\right).$$

which equals,

$$\left(\lim_{h\to 0}g(t+h)\right)\lim_{h\to 0}\left(\frac{1}{h}\left(\mathbf{F}(t+h)-\mathbf{F}(t)\right)\right)+\lim_{h\to 0}\left(\frac{1}{h}\left(g(t+h)-g(t)\right)\right)\mathbf{F}(t)$$

which is just

$$g(t)\mathbf{F}'(t)+g'(t)\mathbf{F}(t)$$

as desired. (And the limit in question exists as well). □

Now we define a new notion. Whether a subset of $\mathbb{R}^2$ (or even $\mathbb{R}$) is "connected".

**Definition 0.7.** We say that a subset $C \subseteq \mathbb{R}^2$ (or in $\mathbb{R}$) is *disconnected* if there are two open subsets, $U$ and $V$, of $\mathbb{R}^2$ such that $U \cap V = \emptyset$, $C \subseteq U \cup V$ and $C \cap U \neq \emptyset$ and $C \cap V \neq \emptyset$.

In words, this means that there are two open subsets of $\mathbb{R}^2$ (or $\mathbb{R}$) such that $C$ has points in common with each of them, every point of $C$ is contained in one of them and the two open sets have no points in common.

**Definition 0.8.** We say that a subset $C \subseteq \mathbb{R}^2$ (or $\mathbb{R}$) is *connected* if it is not disconnected.

The following theorem is just a restatement of what we've already said so far.

**Theorem 0.9.** *A subset $C \subseteq \mathbb{R}^2$ (or in $\mathbb{R}$) is connected if whenever there are two open subsets $U$ and $V$ of $\mathbb{R}^2$ (or $\mathbb{R}$) such that $C \in U \cup V$ and $U \cap V = \emptyset$, then either*

(a) *$C \subseteq U$ or*
(b) *$C \subseteq V$.*

It is a fact that any open interval $(a, b)$ in $\mathbb{R}$ is connected. It also true that $\mathbb{R}$ is a connected subset of itself. You don't need to prove either of these facts. We may assume them for what follows. (The way it is proven typically is by the least upper bound axiom). Intuitively, a subset of $\mathbb{R}^2$ (or $\mathbb{R}$) is connected, if you can get from one point to any other point without the need to "jump".

**Exercise 0.10.** Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is a continuous vector function. Prove that the range of $\mathbf{F}$ is a connected subset of $\mathbb{R}^2$.

*Hint:* Suppose not, then there exist two open subsets $U$ and $V$ of $\mathbb{R}^2$ satisfying various properties. Consider $\mathbf{F}^{-1}(U)$ and $\mathbf{F}^{-1}(V)$.

*Proof.* Suppose that $U$ and $V$ are open sets that show that the range of $\mathbf{F}$ is disconnected. I want to show that $\mathbf{F}^{-1}(U)$ and $\mathbf{F}^{-1}(V)$ give two open sets that prove that $\mathbb{R}$ is disconnected, which contradicts the above fact. So we need to show several things. First note that $\mathbf{F}^{-1}(U)$ and $\mathbf{F}^{-1}(V)$ are open since $\mathbf{F}$ is continuous.

Let us next show that these two sets are disjoint. Suppose $x$ is a real number such that $x \in \mathbf{F}^{-1}(U)$ and $x \in \mathbf{F}^{-1}(V)$. But then $\mathbf{F}(x) \in U$ and $\mathbf{F}(x) \in V$, which is impossible because $U$ and $V$ have no points in common. Therefore, $\mathbf{F}^{-1}(U) \cap \mathbf{F}^{-1}(V) = \emptyset$.

We also need to show that $\mathbf{F}^{-1}(U)$ and $\mathbf{F}^{-1}(V)$ both contain some points of $\mathbb{R}$. But we know that $U$ and $V$ both contain points of the range of $\mathbf{F}$. Say $\mathbf{u} \in U$ is a point of $U$ contained in the range of $\mathbf{F}$. Further suppose that $\mathbf{v} \in V$ is a point of $V$ contained in the range of $\mathbf{F}$. Since $\mathbf{u}$ is in the range of $\mathbf{F}$, there exists $x \in \mathbb{R}$ such that $\mathbf{F}(x) = \mathbf{u}$. Likewise, there exists $y \in \mathbb{R}$ such that $\mathbf{F}(y) = \mathbf{v}$. But then $x \in \mathbf{F}^{-1}(U)$ and $y \in \mathbf{F}^{-1}(V)$, which is what we wanted.

Finally, we need to show that every element of $z \in \mathbb{R}$ is contained in either $\mathbf{F}^{-1}(U)$ or $\mathbf{F}^{-1}(V)$. Since $z \in \mathbb{R}$, we know that $\mathbf{F}(z)$ is contained in the range of $\mathbf{F}$ (by definition). But then either $\mathbf{F}(z) \in U$ or $\mathbf{F}(z) \in V$ since the range of $\mathbf{F}$ is contained in $U \cup V$. This proves that $z$ is either in $\mathbf{F}^{-1}(U)$ or $\mathbf{F}^{-1}(V)$. $\square$

**Exercise 0.11.** Give a brief explanation about why the result of the previous exercise is closely related to the intermediate value theorem.

In the intermediate value theorem, we learned that the range of a continuous function $f : \mathbb{R} \to \mathbb{R}$ can't skip any values. The previous result says the same thing. In this case we showed that the range of a continuous function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is always connected, which means that it can't be broken up in disjoint open sets, which is closely related to the notion of "skipping values"..