MATH 185 – PROBLEMS 1

DUE MONDAY SEPTEMBER 10TH

Question #1.1:

In this question we'll ask fundamental questions about the zero vector $\mathbf{0}$. Which direction does the zero vector really face? Are we sure there is only one zero vector and not an infinite number of different zero vectors facing in different (but infinitely short) directions? Why did we make the assumption that there is a single (aka unique) zero vector?

From one point of view, there might be different ways to answer these questions if one had some alternate exotic theory of vectors. We are just going to talk about vectors in the plane from the geometric point of view assuming what is covered in the text.

(a) Our goal is to give an algebraic reason for their being only one zero vector. We want the rules for vectors $1-30$ through $1-33$ to work for ALL vectors. We also want $=$ to really mean equals. So suppose there is more than one zero vector (in other words, there must be at least two). Give names to two different zero vectors. Explicitly say what these names are in your write-up.

(b) What happens if you add these two *different* zero-vectors together? Apply rule $1-32$ from page 50 of the text (and remember, both of these vectors must be a "zero vector" and must satisfy its properties). What happens if you change the order?

(c) Explain why, if we want rules $1-30$ through $1-33$ to hold, there must be $EXACTLY$ one zero vector.

Once we've decided that we really want their to be exactly one zero vector, it seems unreasonable and possibly even unseemly that it should point in any single direction (what makes one direction "better" than any other?). Thus we say that the zero vector points in every direction equally.

Question #2.2: In this question we'll consider functions which take vectors as input and output other vectors. Recall that a function is just a "rule" that gives a single output which depends (only) on the input. Consider the following functions of vectors.

(i) The function h which takes any vector and $\mathbf{v}$ and halves its magnitude (but leaves its direction alone). That is, $h(\mathbf{v}) = \frac{1}{2}\mathbf{v}$.

(ii) Given any fixed vector $\mathbf{w}$, consider the function $g_{\mathbf{w}}$ which takes an input vector $\mathbf{v}$ and outputs $\mathbf{w} + \mathbf{v}$, that is $g_{\mathbf{w}}(\mathbf{v}) = \mathbf{w} + \mathbf{v}$.

(iii) The function f which takes a vector $\mathbf{v}$ and reverses its direction (but leaves the magnitude alone).

(iv) Given any fixed non-zero vector $\mathbf{w}$, consider the function $L_{\mathbf{w}}$ which takes an input vector $\mathbf{v}$ and outputs a vector with the same direction as $\mathbf{w}$ and the same length as $\mathbf{v}$.

Now for the questions.

(a) Come up with a 5th function that takes an input vector, and then outputs a different vector.

(b) Recall that a function is called *one-to-one* if, whenever you have two different input vectors $\mathbf{v}$ and $\mathbf{v}'$, the associated output vectors are also different. Which of the above functions (including your function) are one-to-one?

(c) Given a function T (which takes vectors as input, and outputs vectors), we say that T is a *linear transformation* if the following two properties hold.

(#1) For any two vectors $\mathbf{v}$ and $\mathbf{v}'$, we always have $T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$. In other words, adding the vectors and then using the function gives the same result as using the function, then adding the two outputs).

(#2) For any $\mathbf{v}$ and any real number c, we have $T(c\mathbf{v}) = cT(\mathbf{v})$. In other words, if you scale the vector and then apply the function, you get the same result as if you'd applied the function and then scaled the output.

Which of the functions above (including the one you created) are linear transformations? Give justification for each answer.