EXTRA PROBLEMS #2

DUE: FRIDAY SEPTEMBER 21

1. Definitions

In this assignment, we'll look at vectors in 3-dimensional space. Vectors in 3-dimensional space are exactly the same as vectors in two dimensional space except they can point in 3 different dimensions. They are still the combined information of a direction and a length. All vectors we consider in this assignment will be vectors in 3-dimensional space. Vectors in 3-dimensional space have the same addition and multiplication properties as vectors in the plane. You can assume that this is known already.

We now define the notion of linearly dependent vectors in 3-dimensional space.

Definition 1.1. If we are considering two vectors $\mathbf{u}$ and $\mathbf{v}$, then we say that they are *linearly dependent* if they are collinear. If there are three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, then we say that they are *linearly dependent* if there are scalars x, y and z (at least one of which is not zero) such that

$$x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}.$$

Definition 1.2. We say that a set of (two or three) vectors is *linearly independent* if the vectors are *NOT* linearly dependent.

Remark 1.3. Note that if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a linearly independent set, then the only solution to the equation

$$x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}.$$

in terms of x, y, and z is when we simultaneously have $x = 0$, $y = 0$ and $z = 0$.

We now define when a set of vectors is a spanning set.

Definition 1.4. A *linear combination* of $\mathbf{u}$,$\mathbf{v}$ and $\mathbf{w}$ is any sum

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$$

where a, b and c are scalars. We say that $\mathbf{u}$,$\mathbf{v}$ and $\mathbf{w}$ is a *spanning set* if every vector $\mathbf{z}$ is equal to a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Finally, we define the notion of a basis.

Definition 1.5. We say that three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is a *basis* if it is a spanning set and it is also linearly independent.

First here are some exercises related to these definitions.

Exercise 1.6. Show that if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a linearly dependent set of vectors, then (at least) one of the vectors is a linear combination of the other two.

Hint: If $a \neq 0$, solve for $\mathbf{u}$, if $b \neq 0$ solve for $\mathbf{v}$, etc.

Exercise 1.7. Justify the following statement: "If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a linearly dependent set, then all three vectors can be realized as line segments contained in the same plane." That is, show that three linearly dependent vectors are coplanar.

Hint: If $\mathbf{z}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, note that $\mathbf{z}$ can be realized as contained in a plane containing both $\mathbf{u}$ and $\mathbf{v}$.

2. The Basis Theorem for vectors in three dimensional space

In this section, you will prove the following theorem.

Theorem 2.1. *Suppose that* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *is a set of three vectors. Then the following statements are equivalent.*

(a) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *is a basis*
(b) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *is a linearly independent set*
(c) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *is a spanning set*
(d) *For every vector* $\mathbf{t}$, *there exists exactly one collection of scalars* x,y *and* z *such that*

$$x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{t}$$

Note that (a) automatically implies (b) and (c) by assumption and that (b) combined with (c) is exactly the definition of (a). We'll first ask you to show that (b) implies (c).

Exercise 2.2. Prove that condition (b) implies condition (c).

Hint: Fix a starting point A and draw all the vectors under consideration as starting at A. Also let P be the end-point of $\mathbf{t}$. Consider the plane determined by $\mathbf{u}$ and $\mathbf{v}$ which also contains P as a point. Extend the vector $\mathbf{w}$ as a line until it intersects this plane (and give a justification as to why it must intersect this plane, use linear independence). Label the point where this line intersects our plane as O. Choose a scalar z so that $z\mathbf{w} = \vec{AO}$, now mimic the first part of the proof of the basis theorem from the book to find the scalars x and y (work in the plane we already constructed and use the linear independence and $\mathbf{u}$ and $\mathbf{v}$).

Remark 2.3. Exercise 2.2 might be the hardest exercise of the set, so if you want to come back to it, that's ok.

Exercise 2.4. Prove that condition (c) implies condition (b).

Hint: Consider what would happen if (b) was not true. Use Exercise 1.7 from the first page. Note that a linear combination of three coplanar vectors gives a fourth vector in the same plane (you may want to draw a picture to justify this).

At this point we have shown that (a), (b) and (c) are all equivalent to each other. We just have to relate (d) to these facts.

Exercise 2.5. Prove that condition (a) implies condition (d).

Hint: Note we have to prove two things. That there is some collection of scalars x,y,z that works (aka existence), and that there is at most one collection of scalars that works (uniqueness). Note that (c) automatically implies existence. To prove uniqueness assume that there are two different solutions in terms of x, y, and z for the same $\mathbf{t}$. Subtract the two equations and then use (b).

Exercise 2.6. Prove that condition (d) implies condition (a).

Hint: Explain why condition (d) implies condition (c).