MATH 185-4, EXAM #2

SOLUTIONS!

1. *The theorem of the mean policeman.* Consider the following situation.

"Jimmy is hungry and decides he wants food at Tim's drive-in which is in a town exactly 30 miles away (in a straight line), we will call this town X. The only road between these two towns has a speed limit of 50mph. He gets in his '57 mustang convertible and drives to town X, picks up lunch at the drive-in and then drives home. The local sheriff notices Jimmy leave town at 12:30pm. He notices Jimmy return at 1:30pm, this time carrying a bag with a half-eaten hamburger from Tim's drive-in. As soon as Jimmy parks his car, the sheriff walks over to Jimmy and writes him a ticket for speeding by 10mph."

Give a mathematically rigorous explanation which proves that the sheriff is correct in assuming that Jimmy was speeding by at least 10mph at some instant. You may assume that the function $J(t)$, which gives the total distance that Jimmy has driven since leaving town, is differentiable. (5 points)

Proof. We assume t is in hours. Note that $J(0) = 0$ and that $J(1) = 60$. Thus $\frac{J(1)-J(0)}{1-0} = 60$. By the "mean" value theorem, there exists a $t_0 \in (0, 1)$ such that $J'(t_0) = \frac{J(1)-J(0)}{1-0} = 60$. This completes the proof. If one wants to argue something about Jimmy's average velocity, and one wants to make it rigorous, one needs to make some kind of argument about integrals or at least areas under curves. □

2. Prove that the function

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$$

is differentiable at $x = 1$. (5 points)

Hint: You may use the fact that a limit exists whenever the left and right limits exist and agree.

Proof. There are a number of correct ways to do this problem. We'll do a short one here. Note that regardless of anything else, we know that f' is defined for $x \neq 1$. We see that

$$f'(x) = \begin{cases} 2x, & x < 1 \\ 2, & x > 1 \end{cases}$$

But then $\lim_{x \to 1^-} f'(x) = 2$ and $\lim_{x \to 1^+} f'(x) = 2$ because polynomials and constant functions are continuous. By a Theorem at the end of section 3-21, we see that f is differentiable at $x = 1$. □

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with a third derivative and suppose that there exists $x_0 \in \mathbb{R}$ so that $f'(x_0) = 0$, $f''(x_0) = 0$ and $f'''(x_0) > 0$. Show that there exists a $\delta > 0$ so that f is increasing on $(x_0 - \delta, x_0 + \delta)$. (10 points)

Hint: What does the graph of f' look like at $x = x_0$ (a graph is not a rigorous proof however).

Proof. Note that $f'(x)$ is concave up at x_0. Thus there exists a $\delta > 0$ such that $f'(x) > f''(x_0)(x - x_0) + f'(x_0)$ for all $x \neq x_0$, $x \in (x_0 - \delta, x_0 + \delta)$. Plugging values in, we see that $f'(x) > 0$ for all $x \neq x_0$, $x \in (x_0 - \delta, x_0 + \delta)$. Since $f'(x_0) = 0$, we see that on the interval $(x_0 - \delta, x_0 + \delta)$, $f'(x) \geq 0$ and furthermore that $f'(x) > 0$ except at a finite number of points. By a theorem in section 3-21, this proves that $f(x)$ is increasing. $\square$

4. Consider the parametric equation

$$x(t) = e^t, y(t) = t^2 + e^t$$

(a) Calculate the value of $\frac{dy}{dx}$ for $t = t_0$. (3 points)

$x'(t) = e^t$, $y'(t) = 2t + e^t$. Thus $\frac{dy}{dx} = \frac{2t+e^t}{e^t}$. At t_0 we get $\frac{dy}{dx}(t_0) = \frac{2t_0+e^{t_0}}{e^{t_0}}$

(b) Write down a formula for the tangent line to the graph at the point $t = t_0$. (2 points)

We know $y = mx + b$, so $y = \frac{2t_0+e^{t_0}}{e^{t_0}}x + b$. The line goes through $(x(t_0), y(t_0))$, so

$$t_0^2 + e^{t_0} = \frac{2t_0 + e^{t_0}}{e^{t_0}}e^{t_0} + b.$$

Solving for b we get

$$b = t_0^2 - 2t_0$$

and so our final equation is

$$y = \frac{2t_0 + e^{t_0}}{e^{t_0}}x + (t_0^2 - 2t_0).$$

(c) What value of t_0 in the interval $[0, 3]$ minimizes the y-intercept of tangent line you just wrote down. Justify your answer (5 points).

Hint: Recall that the y-intercept of a line written in the form $y = mx + b$ is simply the value of b.

We view b as a function of t_0 and so we write $b(t_0) = t_0^2 - 2t_0$. (Note that now t_0 is actually a variable, and not a constant). $b''(t_0) = 2$, so that b is concave up. $b'(t_0) = 2t_0 - 2$ and so we see the only critical point is at $t_0 = 1$. This is in the interval $[0, 3]$ and by the second derivative test, it must be a local min. Therefore, it is the absolute minimum by a theorem from chapter 6.

(EC) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and one-to-one. Further suppose that $f''(x) > 0$ for all $x \in \mathbb{R}$ and that f is a monotonically increasing function and that $f'(x) \neq 0$. Show that f^{-1} is concave down. (5 points)

Proof. Since f is twice differentiable, f' is differentiable, and thus continuous. Therefore $f'(x) \geq 0$ (because if it was < 0, it would be < 0 on some interval, and thus it would be decreasing). But this implies, because $f'(x) \neq 0$, that $f'(x) > 0$.

Now we know that

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

Taking another derivative (using the chain rule) we see that

$$(f^{-1}(x))'' = \left(\frac{1}{f'(f^{-1}(x))}\right)' = \frac{-1}{(f'(f^{-1}(x)))^2}(f''(f^{-1}(x)))(f^{-1}(x))'.$$

Plugging in our formula for $(f^{-1}(x))'$ yet again, we see that

$$(f^{-1}(x))'' = \frac{-1}{(f'(f^{-1}(x)))^2}(f''(f^{-1}(x)))\frac{1}{f'(f^{-1}(x))} = \frac{-f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}$$

Now, f'' of anything is always positive, as is f', so

$$(f^{-1}(x))'' = -(\text{positive})/(\text{positive})$$

and thus $(f^{-1})''$ is negative for all x, so that f^{-1} is concave down. □

(EC) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. Further suppose that $f''(x) < 0$ for all $x \in \mathbb{R}$. Prove that for each real number L, the equation $L = f(x)$ has at most two solutions. (5 points)

Hint: Analyze the solutions to the equation $f'(x) = 0$.

Proof. Suppose that $L = f(x)$ has at least three different solutions, we will prove a contradiction. Let us label $x_1 < x_2 < x_3$ as three of the solutions. Since $f(x_1) = f(x_2)$, by Rolle's theorem, we see that there exists a $x_4 \in (x_1, x_2)$ such that $f'(x_4) = 0$. Likewise, there exists $x_5 \in (x_2, x_3)$ such that $f'(x_5) = 0$. Now, since $f''(x) < 0$, $f'(x)$ is monotonically decreasing and thus f' is one to one. However, $f'(x_4) = f'(x_5)$ (since they are both zero) and $x_4 < x_2 < x_5$ which demonstrates that $x_4 \neq x_5$, which is a contradiction.

Remark 0.1. It is possible to do a careful and correct by first showing that f' can cross zero at most once. Furthermore, the sign of f' changes only at that point (if it exists) because f' is continuous (essentially by the intermediate value theorem). Then one can analyze the cases where $f' < 0$ and $f' > 0$ individually, and make the same conclusion.

□