The `TestIdeals` package for Macaulay2

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- Consider rings R of characteristic $p > 0$.
- No resolution of singularities (in general).
- Kunz proved:

Theorem (Kunz)

R is regular if and only if Frobenius is flat.

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- How flat is Frobenius?

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History, part 2

- Because we are working with computers, domain finite type over $\mathbb{F}_q$.
- Kunz says Frobenius is flat if and only if R^{1/p^e} is locally free over R.
- We can weaken being locally free.

Definition (Hochster-Roberts, Mehta-Ramanathan)

R is *F-pure* if and only if $R \to R^{1/p^e}$ splits.

- F-pure is analogous to log canonical singularities.
  - F-pure implies SLC.
  - SLC in char 0 conjecturally implies F-pure for many p.

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Checking F-purity can be pretty easy.

- Fedder's Criterion. $R = S/I$, S is polynomial.

Theorem (Fedder)

R is F-pure at $\mathfrak{m}$ if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$.

- If $I = (f)$, then $I^{[p]} : I = (f^{p-1})$. (*BOARD*)
- For example.

```
i5 : S = ZZ/7[x,y,z];
i6 : f = x^3 + y^3 + z^3;
i8 : isSubset(ideal(f^6), ideal(x^7, y^7, z^7))
o8 = false
```

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We have written a package `TestIdeals.m2` that computes whether a ring (or pair) is:

- *F*-pure
  - Analog of SLC.
- *F*-regular
  - Analog of KLT.
- *F*-rational
  - Analog of rational.
- *F*-injective
  - Analog of Du Bois.
- Test ideals
  - Analogs of multiplier ideals
- *F*-pure thresholds (with `FThresholds.m2`).
  - Analogs of log canonical thresholds.

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Macaulay2

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Fedder, part 2

- Fedder's criterion works because maps

$$\phi_R : R^{1/p^e} \to R$$

come from maps

$$\phi_S : S^{1/p^e} \to S$$

such that $\phi_S(I^{1/p^e}) \subseteq I$.

- In fact,

$$I^{[p^e]} : I \cong \{\phi \in \mathrm{Hom}_S(S^{1/p^e}, S) \mid \phi(I^{1/p^e}) \subseteq I\}.$$

- Translates questions on R into questions in polynomial ring S.
- Note $\{\phi_R \neq 0\} \leftrightarrow \{\Delta \geq 0 \ \ \mathbb{Q}\text{-log boundary}\}$. (*BOARD*)

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Frobenius trace

One more big tool.

- There exists $\Phi : S^{1/p^e} \to S$.
- $\Phi\left(x_1^{\frac{p^e-1}{p^e}} \cdots x_1^{\frac{p^e-1}{p^e}}\right) = 1$
- Other monomials to 0.
- Φ generates $\mathrm{Hom}_S(S^{1/p^e}, S)$.
- Φ is Grothendieck dual to Frobenius.
- $\Phi(J^{1/p^e}) \subseteq I$ if and only if

$$I^{[p^e]} \subseteq J.$$

Theorem (Fedder restated)

$$\Phi((I^{[p^e]} : I)^{1/p^e}) \equiv_I \mathrm{Image}(\mathrm{Hom}_R(R^{1/p^e}, R) \xrightarrow{@1} R)$$

defines locus where $R = S/I$ is not F-pure.

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We compute some Macaulay2 examples. $\Phi(J)$ is called the *Frobenius root of J*.

```
i12 : I =ideal(x^3 + y^3 + z^3);
i13 : frobeniusRoot(1, I^7 : I)
o13 = ideal 1
i14 : isFPure(S/I)
o14 = true
i15 : J = ideal(x^4+y^4+z^4);
i16 : frobeniusRoot(1, J^7 : J)
                 2            2        2
o16 = ideal (z , y*z, x*z, y , x*y, x )
i19 : isFPure(S/J)
o19 = false
```

More examples

```
i20 : T = ZZ/5[a,b,c,d,e];
i21 : B = ZZ/5[x,y];
i22 : f = map(B, T, {x^4, x^3*y, x^2*y^2, x*y^3, y^
                   4    3     2  2       3    4
o22 = map(B,T,{x , x y, x y , x*y , y })
o22 : RingMap B <--- T
i23 : I = ker f
                 2                               2
o23 = ideal (d  - c*e, c*d - b*e, b*d - a*e, c  - a
o23 : Ideal of T
i24 : isFPure(T/I)
o24 = true
```

F-regularity and test ideals

- Analog of KLT.

Definition

R is *strongly F-regular* if for every (interesting[a]) $c \in R$, there is some e and $\phi : R^{1/p^e} \to R$ so that $\phi(c^{1/p^e}) = 1$.

[a] In Jacobian ideal is good enough

- If translated by Fedder's methods,

Theorem

$R = S/I$ is strongly F-regular if and only if

$$I + \Phi((c(I^{[p^e]} : I))^{1/p^e}) = S.$$

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F-regularity checking

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i4 : R = S/ideal(x^2-y*z)
i5 : isFRegular(R);
o5 = true
i20 : A = ZZ/7[x,y,z]/(y^2*z - x*(x-z)*(x+z));
i21 : C = ZZ/7[a,b,c,d,e,f];
i22 : g = map(A, C, {x^2, x*y, x*z, y^2, y*z, z^2})
i23 : I = ker g;
i26 : isFRegular(C/I);
o26 = false
```

- We can only show that $\mathbb{Q}$-Gorenstein rings are not F-regular.
- The `QGorensteinIndex=>infinity` option can prove a non-$\mathbb{Q}$-Gorenstein ring is F-regular.

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F-regularity of pairs

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i6 : h = y;
i7 : isFRegular(1/2, y)
o7 = false
i8 : isFRegular(1/3, y)
o8 = true
```

- The pair $(R, h^{1/2})$ is not F-regular but $(R, h^{1/3})$ is.
- The `FThresholds` package can even compute F-pure thresholds.

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o7 = false
i8 : isFRegular(1/3, y)
o8 = true
```

- The pair $(R, h^{1/2})$ is not F-regular but $(R, h^{1/3})$ is.
- The `FThresholds` package can even compute F-pure thresholds.

- Analog of rational singularities.
- Implies (pseudo-)rational singularities in a fixed characteristic.
  - $\mathcal{O}_X \simeq R\pi_*\mathcal{O}_Y$
- Here's our definition:

Definition

R has *F-rational singularities* if it is

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Implementation

The tricky part is writing the map:

$$F - \text{dual} : \omega_{R^{1/p^e}} \to \omega_R.$$

- Trick (Katzman) is to embed ω_R as an ideal in R.
- Extend $F - \text{dual}$ to $\phi_R : R^{1/p^e} \to R$.
- Extend further to $\phi_S : S^{1/p^e} \to S$. $(R = S/I)$
- Represent $\phi_S \in I^{[p^e]} : I$.

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F-rational examples

Here is an example of an *F*-rational (but not *F*-regular) ring.

```
i8 : S = ZZ/3[a,b,c,d,t]; m = 4; n = 3;
i11 :  M = matrix{ {a^2 + t^m, b, d},
                   {c, a^2, b^n-d} };
                  2        3
o11 : Matrix S  <--- S
i12 :  I = minors(2, M);
i13 : R = S/I;
i14 : isFRational(R)
o14 = true
```

Appeared in work of Anurag Singh (deform *F*-regularity)

Characteristic zero applications

Characteristic $p > 0$ conclusions imply results in characteristic zero.

Theorem (Ma-•)

Suppose R is a ring of mixed characteristic finite type over $\mathbb{Z}$. Suppose $p \in \mathbb{Z}$ is a regular element and $Q \subseteq R$ is a prime not containing any nonzero prime of $\mathbb{Z}$ so that $(p) + Q \neq R$.

If R/pR is F-rational, then $R_Q = R_Q \otimes \mathbb{Q}$ has rational singularities.

- Analogous statement for log terminal/F-regular singularities, if the $\mathbb{Q}$-Gorenstein not divisible by p.
- Not known for log canonical/F-pure singularities (need mixed char inversion of adjunction).

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F-injective

We can also study F-injective singularities (analog of Du Bois).

Definition

R is *F-injective* if

$$H^{-i}\omega^{\bullet}_{R^{1/p}} \to H^{-i}\omega^{\bullet}_{R}$$

surjects for all i.

- If R is CM, this just means

$$(\omega_{R^{1/p^e}}) \xrightarrow{F^e-\text{dual}} \omega_R$$

surjects.

- Example

```
i10 : R = ZZ/[x,y,z]/ideal(x^3+y^3+z^3);
i11 : isFInjective(R)
o11 = true
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Test ideals

We can compute test ideals too. Including of pairs.

- In a $\mathbb{Q}$-Gorenstein ring.
- $\tau(R, f^t)$ equals sum of images of maps

$$\phi : (cf^{\lceil t(p^e-1)\rceil} R)^{1/p^e} \to R.$$

  c as before.
- We use it to check F-regularity.
  - (R, f^t) is F-regular if and only if $\tau(R, f^t) = R$.
- Trick is stabilize image sums above.
- Can compute parameter test modules and parameter test ideals too.

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Example

```
i2 : R = ZZ/5[x,y];
i3 : f = y^2-x^3;
          3     2
o3 = - x   + y
i4 : testIdeal(4/5, f);
o4 = ideal (y, x)
i5 : testIdeal(4/5-1/10000, f)
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```

- We can compute $\tau(R, f^{t-\epsilon})$, which is used to compute jumping numbers and F-pure thresholds.
- Needs `HSLGModule` function.

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F-pure thresholds

We conclude with a discussion of the `FThresholds` package.

- If R is F-regular, F-pure threshold (FPT) is the smallest $t \geq 0$ where $\tau(R, f^t) \neq R$.
- We do a binary-style search to a certain depth.
- However, if f is a special form, we have other algorithms.
- We also guess + check.

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```
i2 : R = ZZ/5[x,y,z]
i3 : f = x^5 - y^6 + x^3*z^5 + 2*z^8
      3 5      8     6     5
o3 = x z  + 2z  - y  + x
i4 : fpt(f)
     1
o4 = -
     5
```

FPT of the cusp (in a nonstandard form).

```
i1 : R = ZZ/7[x,y]
i4 : fpt((x+y)^3 - y^2)
     5
o4 = -
     6
```

Thanks!

You can go to:

http://www.math.utah.edu/~schwede/M2.html

to try it yourself!