Game 2
Computational Science Battery $\bigoplus$ Math

What follows is a series of puzzles on which to test your newfound coding ability. Using what you have learned, you will attempt to solve these puzzles to earn points for your team. Each correctly solved puzzle earns your team 20 points, but an incorrect solutions loses your team 5 points.

These puzzles are separated into two categories:

Propositions

Each proposition is a claim that something is true. Some of the propositions are in fact true, some are false and some are open problems (nobody knows the answer). You may solve these puzzles in one of two ways: prove that the statement is true or find a counterexample that proves it's false. No points are awarded if you say a statement is an open problem.

Computations

These puzzles ask you to produce a number that is the answer to a question. The idea is that you write a program to produce the solution, though if you can produce a solution without using the computer that is also allowed. You may, and should, use any code that you have produced in the labs thus far.

> Note that neither the propositions nor the computations are listed in order of increasing difficulty, so you should look through all of them and find the ones that look the most approachable to you.

Rules

- You may not at any time look for solutions to any of the propositions or the computations online or use any code that has not been produced by yourself or your teammates.
- You may ask a TA to help you figure out why you are getting errors at a cost of 5 points. If you are not getting errors but your code is not working properly, you may ask a TA to help you figure out why at a cost of 10 points. At no time will a TA tell you how to do a problem. If you do not understand a puzzle, you may ask for clarification free of charge.

Propositions

1. **Pascal's Triangle:** If p is prime, then every nontrivial (i.e. exclude the 1's at either end) element of the pth row of Pascal's triangle is divisible by p.

2. **Pascal's Triangle the other direction:** If $n > 1$ and every nontrivial element of the nth row of Pascal's triangle is divisible by n, then n is prime.

3. **Converse of Fermat's Little Theorem:** If $a^{n-1} \equiv 1 \bmod n$ for every $a < n$ with $\gcd(a, n) = 1$, then n is prime.

4. **Totient Property:** Euler's totient function $\varphi(n)$ is always even when n is at least 3.

5. **Another Totient property:** Notice that 12 is divisible by 1, 2, 3, 4, 6 and 12. Also,

$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 1 + 1 + 2 + 2 + 2 + 4 = 12$$

   More generally, for every n it is the case that n is equal to the sum of $\varphi(d)$ for all d that evenly divide n, including 1 and n itself.

6. **Totient Order Divisibility:** When working mod n, the **first** power k of a number x that is congruent to 1 (so $x^k \equiv 1 \bmod n$) is called the order of x mod n. For example, 2^3 is congruent to 1 mod 7, so we say 2 has order three mod seven. If x^k is never 1 mod n then we say x has infinite order.

   If the order of x mod n is not infinite, then the order of x evenly divides $\varphi(n)$.

7. **Goldbach Conjecture:** Any even number larger than two is the sum of two primes.

8. **Fermat's Claim:** For any positive integer n, $2^{2^n} + 1$ is prime.

9. **Sort-Of-Converse of Fermat's Claim:** If $2^k + 1$ is prime and k is not equal to zero, then $k = 2^n$ for some integer n.

10. **Collatz Conjecture:** Choose some number a_0. Define a sequence $\{a_n\}$ by

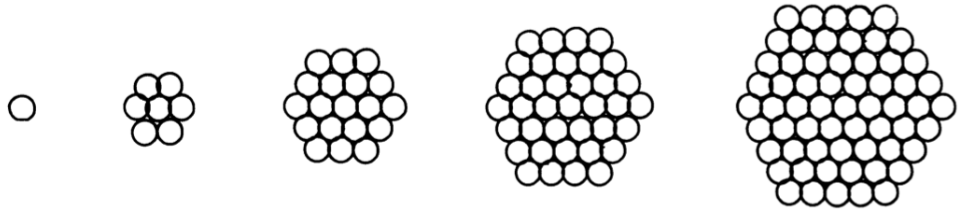
$$a_n = \begin{cases} 3a_{n-1} + 1 & \text{if } a_{n-1} \text{ is odd} \\ \frac{a_{n-1}}{2} & \text{if } a_{n-1} \text{ is even} \end{cases}$$

   Then a_n will be 1 for some n.

11. **The Infinitude of Primes:** There are infinitely many primes that are one more than a multiple of 4. There are also infinitely many primes that are one less than a multiple of 4.

12. **The Prime Number Race:** For any number n, the list of primes smaller than n contains at least as many primes of the form $4n - 1$ as of the form $4n + 1$.

13. **Partial Sums of Hex Numbers are Perfect Cubes:** Hex numbers can be arranged in a hexagon as shown below. The sum of the first n hex numbers is always a perfect cube.



1 7 19 37 61

14. **nth Powers of 2 mod n** It is never true that 2^n is congruent to 1 mod n. It is also never true that 2^n is congruent to 3 mod n.

Computations

1. Triangle numbers are given by the formula $T_n = \frac{n(n+1)}{2}$.

   Pentagonal numbers are given by the formula $P_n = \frac{n(3n+1)}{2}$

   Hexagonal numbers are given by the formula $H_n = n(2n-1)$.

   Find a number greater than 1 that is a triangle, a pentagon and a hexagon simultaneously.

2. What is the smallest triangle number that is evenly divisible by at least 500 different numbers, including 1 and itself?

3. What is the smallest number that is evenly divisible by all of the numbers from 1 to 20?

4. What do you get when you add all of the digits in 100! ?

5. A Pythagorean triple is a triple of integers a, b, c where $a^2 + b^2 = c^2$. How many Pythagorean triples are there where a, b and c are smaller than 1000?

6. Find three squares in an arithmetic progression. That is, find a, b, c such that $a^2 < b^2 < c^2$ and the difference between a^2 and b^2 is the same as the difference between b^2 and c^2.

7. Determine how many triples of squares are in arithmetic progression as in the previous problem when a, b and c are all less than or equal to 100.

8. Observe that 65 can be written as a sum of two squares in two different ways; $65 = 8^2 + 1^2 = 4^2 + 7^2$. Find another number that can be written as a sum of two squares in two different ways and what two pairs of squares add to it.

9. What do you get when you add all of the prime numbers less than one-million?

10. Suppose you add $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{100}$ and write the answer as a reduced fraction. What is the numerator?

11. Proposition 9 on the previous page defined how to generate a Collatz sequence from a starting value. Which starting value less than 100 produces the longest Collatz sequence before hitting 1 for the first time?

12. Which term of the Fibonacci sequence is the first to contain at least 1000 digits? Give the place in the sequence, not the number itself.

13. What do you get when you add all of the palindrome numbers less than 1000?

14. Observe that the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, and that 1+ 2+ 4+ 5+ 10+ 11+ 20+ 22+ 44+ 55 +110 = 284.

    Observe further that the proper divisors of 284 are 1, 2, 4, 71 and 142, and that 1+2+4+71+142 = 220.

    Find two other numbers with this property.

15. What is the 1000th digit of the decimal expansion for $\sqrt{2}$?

16. You may have heard that the arithmetic mean of two numbers is always at least as large as the geometric mean;
$$\frac{a+b}{2} \geq \sqrt{ab}$$
If a and b are different positive numbers less than 1000 and neither is zero, what is the largest possible difference between the arithmetic and geometric averages of a and b?

17. How many ways are there to make \$1.00 in change? By change we mean pennies, nickels, dimes and quarters; no fifty-cent pieces or dollar coins.

18. Observe $2+3+5+7+11+13 = 41$. Notice that $2, 3, 5, 7, 11, 13$ are consecutive primes and that 41 is also prime. Find the largest number below 1000 that is prime and is the sum of consecutive primes.

19. Six people are randomly given either a black or white hat and they are arranged so that everyone can see everyone else, but nobody can see themselves. Simultaneously, they must all say what color their hat is, or pass. A single correct guess saves them all, while a single incorrect guess is failure for them all.

    Their strategy is for each person to guess the opposite color of the color they see most. What is the probability of this strategy working?