1. A proof of a version of Schäfer's Theorem

In 1955 (Über die Methode der a priori Schranken, Math. Ann. 129(1955), 415-416) H. Schäfer outlined a dichotomy about solutions of the equation

$$u - \lambda F(u) = 0, \tag{1.1}$$

where

$$F : E \to E,$$

is a completely continuous mapping from the real Banach space E, with norm $\|\cdot\|$ to itself. We shall establish here a version which immediately follows from properties of the Leray-Schauder degree, namely the following:

Theorem 1.1. *Let*

$$S^+ = \{(\lambda, u) : u = \lambda F(u)\},$$

then S^+ is unbounded in $[0, \infty) \times E$. More precisely the following hold:

1. *If there exists $\lambda_0 > 0$ such that for $\lambda = \lambda_0$ (1.1) has no solution u, then for every $R > 0$, there exists $\lambda_R \in (0, \lambda_0)$ such that (1.1) has a solution for $\lambda = \lambda_R$ with $\|u_R\| = R$.*
2. *If there exists $R_0 > 0$ such that for $R = R_0 > 0$ (1.1) does not have a solution u for any $\lambda > 0$, then for every $\lambda > 0$, (1.1) has a solution u_λ with $\|u_\lambda\| < R_0$.*

An analagous result holds for $\lambda \in (-\infty, 0]$.

Proof: Let us consider the first alternative. If there exists $R > 0$ such that (1.1) has no solutions u with $\|u_R\| = R$, for any $\lambda \in (0, \lambda_0]$, then by the homotopy invariance principle of the Leray-Schauder degeree we must have that

$$\mathrm{d}(\mathrm{id} - \lambda F, B_R, 0) = \text{constant}, \ 0 \le \lambda \le \lambda_0.$$

This, however cannot hold, since

$$\mathrm{d}(\mathrm{id}, B_R, 0) = 1, \ \mathrm{d}(\mathrm{id} - \lambda_0 F, B_R, 0) = 0.$$

In the above, B_R is the open ball in E of radius R centered at the origin.

Let us consider the second alternative. If there exists $R_0 > 0$ such that (1.1) has no solutions u with $\|u\| = R_0$, for any $\lambda \in (0, \infty)$, then by the homotopy invariance principle of the Leray-Schauder degeree we must have that

$$\mathrm{d}(\mathrm{id} - \lambda F, B_{R_0}, 0) = \text{constant}, \ 0 \le \lambda \le \mu,$$

where $\mu > 0$ is arbitrary. Since

$$\mathrm{d}(\mathrm{id}, B_{R_0}, 0) = 1,$$

this degree is nonzero; hence by the solution property we have that for all such λ, (1.1) has a solution u_λ with $\|u_\lambda\| < R_0$.

It is instructive to reconsider example 14, page 18, in this light, and observe that the boundary value problem considered there will have an unbounded set of solutions with the values of λ contained in $(0, 1]$.