Exercise 1.2.3 Let $n$ be an integer. Then $n$ is either even or odd. If $n$ is even, then $n = 2k$, where $k$ is an integer. Then $n^2 = 4k^2$, which is divisible by 4. If $n$ is odd, then $n = 2k + 1$, where $k$ is an integer. Hence $n^2 = 4k^2 + 4k + 1$, which, when divided by 4, leaves a remainder of 1.

Exercise 1.2.4 If both $a$ and $b$ were odd, then $a^2 = 4k^2 + 4k + 1$ and $b^2 = 4l^2 + 4l + 1$, where $k$ and $l$ are integers, hence

$$a^2 + b^2 = 4(k^2 + l^2) + 4(k + l) + 2 = 4(k^2 + l^2 + k + l) + 10,$$

which when divided by 4 yields a remainder of 2. On the other hand, if $(a, b, c)$ is a Pythagorean triple, then

$$a^2 + b^2 = c^2,$$

and, by the previous exercise $c^2$ will leave a remainder of 0 or 1, when divided by 4. This is in contradiction to the fact that $a^2 + b^2$, when divided by 4 leaves a remainder of 2. Thus, the assumption that both $a$ and $b$ are odd cannot be correct, and either $a$ or $b$ must be even.

Exercise 1.3.1 If $(a, b, c)$ is a Pythagorean triple, then, as was verified before (or by definition of Pythagorean triple) we have

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1,$$

with $\frac{a}{c}$ and $\frac{b}{c}$ rational numbers. By what has been proved on pages 6 and 7, we must have that

$$\frac{a}{c} = \frac{1 - t^2}{1 + t^2}, \quad \frac{b}{c} = \frac{2t}{1 + t^2},$$

where $t = \frac{p}{q}$ is a rational number, i.e., $p$ and $q$ are integers. Substituting this expression into (1) and simplifying, one obtains the desired expression.

Exercise 1.3.2 Since

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad \frac{b}{c} = \frac{2pq}{p^2 + q^2},$$

for some integers $p, q$, it follows that

$$a = \frac{p^2 - q^2}{p^2 + q^2}c, \quad b = \frac{2pq}{p^2 + q^2}c.$$

Since $a$ and $b$ are both integers and since $\frac{p^2 - q^2}{p^2 + q^2}$ and $\frac{2pq}{p^2 + q^2}$ are fractions whose absolute value is less than 1, it follows that $c$ must be a multiple of $p^2 + q^2$, say $c = r(p^2 + q^2)$. This implies the assertion.
Exercise 1.3.3 Any right triangle with hypotenuse 1 may be obtained from Figure 1.4. The point $P$ in this figure, on the other hand may be assigned coordinates

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right),$$

where $t$ is the slope of the line defined in Figure 1.5. If the point $P$ does not have rational coordinates, the the slope $t$ will not be rational. On the other hand, there will be rational numbers arbitrarily close to $t$. Such slopes will yield triangles (of hypotenuse 1) which are arbitrarily close in area to the area of the original triangle. To note, the area of a given triangle, generated using the construction of the text with slope $t$ is given by

$$A(t) = \frac{1-t^2}{1+t^2} \cdot \frac{2t}{1+t^2}.$$ 

Note that this is a continuous function of $t$, hence for a given irrational $t_0$ and a given accuracy, we can find rational values of $t$ arbitrarily close to $t_0$ such that $A(t)$ will be within the given accuracy of $A(t_0)$.

Exercise 1.4.1 Let $a$ denote the length of the side of the large solid squares and $b$ that of the small solid square. The length of the side of the dotted squares will be the length of the right triangle with smaller legs $a$ and $b$. Marking triangles appropriately (here it would be best to use a coloring scheme and a lot of words to explain it), one will find that $a^2 + b^2 = c^2$.

Exercise 1.4.2 Let

$$c = c_1 + c_2,$$

and denote by $h$ the height of the large right triangle, indicated in the figure by the solid vertical line segment. By the laws of similar triangles, we have the following proportions:

$$\frac{c_1}{a} = \frac{a}{c},$$

i.e. $a^2 = cc_1$, and

$$\frac{c_2}{b} = \frac{b}{c},$$

i.e. $b^2 = cc_2$, and thus

$$a^2 + b^2 = cc_1 + cc_2 = c(c_1 + c_2) = c^2.$$