

1. EXERCISES-DEGREE OF MAPS 2-SOLUTIONS

- (1) Let A be an $n \times n$ real matrix for which 1 is not an eigenvalue. Let Ω be a bounded open neighborhood of $0 \in \mathbb{R}^n$. Show, using linear algebra methods, that

$$d(\text{id} - A, \Omega, 0) = (-1)^\beta,$$

where β equals the sum of the algebraic multiplicities of all real eigenvalues μ of A with $\mu > 1$.

Solution: We recall from the basic definition of the Brouwer degree that

$$d(\text{id} - A, \Omega, 0) = \text{sgndet}(\text{id} - A),$$

on the other hand

$$\det(\text{id} - A) = \prod \{\lambda : \lambda \text{ an eigenvalue of } (\text{id} - A)\}$$

with eigenvalues repeated in the above product as often as their algebraic multiplicity, and λ is an eigenvalue of $(\text{id} - A)$ if and only if $\lambda = 1 - \mu$ where μ is an eigenvalue of A . Furthermore, since complex eigenvalues appear in conjugate pairs, they will not contribute to the sign of the above determinant, as won't the positive eigenvalues. Thus, to determine the sign of the above product we only need to consider negative λ , or equivalently eigenvalues μ of A which are bigger than 1.

- (2) Let $\Omega \subset \mathbb{R}^n$ be a symmetric bounded open neighborhood of $0 \in \mathbb{R}^n$ and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $0 \notin f(\partial\Omega)$. Also assume that

$$\frac{f(x)}{|f(x)|} \neq \frac{f(-x)}{|f(-x)|}, \quad x \in \partial\Omega.$$

Show that $d(f, \Omega, 0)$ is an odd integer.

Solution: We observe that for any continuous function f the mapping

$$x \mapsto f(x) - f(-x)$$

is an odd mapping. Consider the function

$$h(t, x) = f(x) - tf(-x), \quad t \in [0, 1], \quad x \in \bar{\Omega}.$$

We note that if there exists $t \in (0, 1]$ such that

$$h(t, x) = 0,$$

for some $x \in \partial\Omega$, we conclude that for this value of x

$$\frac{f(x)}{|f(x)|} = \frac{f(-x)}{|f(-x)|}, \quad x \in \partial\Omega,$$

contradicting the assumption. Thus, by the homotopy invariance principle we have that

$$d(h(0, \cdot), \Omega, 0) = d(h(1, \cdot), \Omega, 0),$$

i.e.

$$d(f, \Omega, 0) = d(f(\cdot) - f(-\cdot), \Omega, 0),$$

the latter, on the other hand is an odd mapping, and since Ω is a symmetric neighborhood of the origin, it follows from Borsuk's theorem that the degree is an odd integer.

- (3) Let Ω be as in Exercise 9 and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be an odd function such that $f(\partial\Omega) \subset \mathbb{R}^m$, where $m < n$. Show there exists $x \in \partial\Omega$ such that $f(x) = 0$.

Solution: Let us assume that f has no zeros in $\partial\Omega$. It follows from properties of the degree that

$$d(f, \Omega, 0) = d(h, \Omega, 0),$$

for any continuous mapping $h : \bar{\Omega} \rightarrow \mathbb{R}^n$ having the property that

$$h(x) = f(x), \quad x \in \partial\Omega.$$

(See Corollaries 16 and 17.) We are given that

$$f : \partial\Omega \rightarrow \mathbb{R}^m, \quad m < n.$$

Since $\partial\Omega$ is a closed set and \mathbb{R}^m is a convex set, it follows from Theorem 20, page 23, that there exists a continuous extension

$$\bar{f} : \bar{\Omega} \rightarrow \mathbb{R}^m, \quad m < n$$

which agrees with f on $\partial\Omega$. Hence

$$d(\bar{f}, \Omega, 0) = d(f, \Omega, 0).$$

On the other hand the latter is an odd integer, since f is an odd mapping. It follows from the continuity property of the degree (Proposition 13, page 39) that there exists $\epsilon > 0$, such that

$$d(\bar{f} - y, \Omega, 0) = d(\bar{f}, \Omega, 0),$$

for any $y \in \mathbb{R}^n$, $|y| < \epsilon$, which, of course implies that for all such y the equation

$$\bar{f}(x) = y,$$

has a solution in Ω , i.e. $\bar{f}(\Omega)$ contains the ball of radius ϵ , centered at the origin, in \mathbb{R}^n contradicting that the range of \bar{f} is contained in \mathbb{R}^m , $m < n$.

- (4) Let f and Ω be as in the previous exercise, except that f is not necessarily odd. Show there exists $x \in \partial\Omega$ such that $f(x) = f(-x)$.

Solution: Apply the previous exercise with $f(x)$ replaced by $f(x) - f(-x)$, which is an odd function.