1. Exercises-Implicit functions-degree of maps

1.1. **Exercise.** Consider the forced nonlinear oscillator (periodic boundary value problem)

$$u'' + \lambda u + u^2 = g,\ u(0) = u(2\pi),\ u'(0) = u'(2\pi) \tag{1}$$

where g is a continuous $2\pi - \textit{periodic}$ function and $\lambda \in \mathbb{R}$, is a parameter. Let $E = C^2([0, 2\pi], \mathbb{R}) \cap \{u : u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}$, and $X = C^0([0, 2\pi], \mathbb{R})$, where both spaces are equipped with the norms discussed earlier. Then for certain values of λ, (1) has a unique solution for all forcing terms g of small norm.

Solution: We shall demonstrate the conclusion by a use of the Implicit Function Theorem. To this end we shall use

$$\Lambda = \{(\lambda, g) : \lambda \in \mathbb{R}, u \in X\}$$

with

$$\|(\lambda, u)\|_\Lambda = |\lambda| + \|u\|_0.$$

We define the mapping

$$f : E \times \Lambda \to X$$

by

$$f(u, \lambda) = u'' + \lambda u + u^2 - g$$

and look for zeros of this mapping. Note that this is a continuous mapping. To see this, let (λ_0, g_0, u_0) be an arbitrary point in $\Lambda \times E$ and let $\epsilon > 0$ be given. We compute

$$\begin{aligned} \|f(u, \lambda) - f(u_0, \lambda_0)\|_0 \quad &\leq \quad \|u'' - u_0''\|_0 + |\lambda - \lambda_0|\|u_0\|_0 \\ &\quad + |\lambda_0|\|u_0\|_0 + \|u_0 + u\|_0\|u_0 - u\|_0 + \|g - g_0\|_0. \end{aligned}$$

Then, if

$$|\lambda - \lambda_0| + \|g - g_0\|_0 + \|u - u_0\|_2 \leq \delta \leq 1$$

we obtain

$$\begin{aligned} \|f(u, \lambda, g) - f(u_0, \lambda_0, g_0)\|_0 \quad &\leq \quad \|u - u_0\|_2 + |\lambda - \lambda_0|\|u_0\|_2 \\ &\quad + |\lambda_0|\|u - u_0\|_2 \\ &\quad + (\|u_0\|_2 + 1)\|u_0 - u\|_2 + \|g - g_0\|_0, \end{aligned}$$

or

$$\begin{aligned} \|f(u, \lambda, g) - f(u_0, \lambda_0, g_0)\|_0 \quad &\leq \quad \delta + \delta\|u_0\|_2 \\ &\quad + |\lambda_0|\delta + (\|u_0\|_2 + 1)\delta + \delta \\ &= (2 + 2\|u_0\|_2 + |\lambda_0|)\delta. \end{aligned}$$

Hence, choosing

$$\delta \leq \min\{1, \frac{\epsilon}{2 + 2\|u_0\|_2 + |\lambda_0|}\},$$

we have

$$\|f(u, \lambda, g) - f(u_0, \lambda_0, g_0)\|_0 \leq \epsilon.$$

Thus f is a continuous mapping.

We next compute

$$f(u + h, \lambda, g) - f(u, \lambda, g) = h'' + \lambda h + 2uh + h^2.$$

On the other hand

$$\frac{\|h^2\|_0}{\|h\|_2} \leq \frac{\|h\|_2^2}{\|h\|_2},$$

thus

$$h^2 = o(\|h\|_2).$$

This proves that

$$D_u f(u, \lambda, g)(h) = h'' + (\lambda + 2u)h$$

and

$$\|D_u f(u, \lambda, g)(h)\|_0 \leq (1 + \|\lambda + 2u\|_2)\|h\|_2.$$

Thus $D_u f(u, \lambda, g)$ is a bounded linear map for each (λ, g). Note that

$$\|D_u f(u, \lambda, g)(h) - D_u f(v, \mu, g)(h)\|_0 \leq (|\lambda - \mu| + \|u - v\|_2)\|h\|_2$$

from which one may conclude that the mapping

$$(u, \lambda, g) \mapsto D_u f(u, \lambda, g)$$

is a continuous mapping from $E \times \Lambda$ to the bounded linear maps from E to X.

We next note that

$$f(0, \lambda, 0) = 0$$

and

$$D_u f(0, \lambda, 0)(h) = h'' + \lambda h.$$

If we let λ range over the values n^2, then solutions of

$$h'' + n^2 h = 0$$

are given by

$$h(x) = a \sin nx + b \cos nx,$$

which are all periodic of period 2π for any values of a and b. Hence for those values of λ we cannot apply the implicit function theorem (the operator is not one to one). Let us therefore fix $\lambda \neq n^2,\ n = 0, \pm 1, \cdots$. In this case all solutions of

$$h'' + \lambda h = g \tag{2}$$

are obtained by the variation of constants method and one may show Using this method) that there is a unique solution h which resides in the space E. Let us carry out this procedure in a particular case that λ_0 is a poitive number $\lambda_0 \neq n^2,\ n = 0, \pm 1, \cdots$. Then the general solution of (2) is

$$h(x) = a \sin \sqrt{\lambda_0} x + b \cos \sqrt{\lambda_0} x + h_p(x), \tag{3}$$

where

$$h_p(x) = \frac{1}{\sqrt{\lambda_0}} \int_0^x (\sin \sqrt{\lambda_0} x \cos \sqrt{\lambda_0} t - \sin \sqrt{\lambda_0} t \cos \sqrt{\lambda_0} x) g(t) dt.$$

The constants a and b in (3) are uniquely determined by the boundary conditions (do the algebra!) and after calculating them, one may show that (again you may want to do the algebra)

$$\|h\|_2 \leq c\|g\|_0.$$

Hence $D_u f(0, \lambda_0, 0)$ is a linear homeomorphism of E onto X. We may therefore apply the implicit function theorem to show that for all λ close to λ_0 and all $g \in X$ whose norm is small, there exists a unique solution $u(\lambda, g) \in E$ of small norm of our problem.

1.2. **Exercise.** Let Ω be a bounded open set in $\mathbb{C}$ and let $f : \overline{\Omega} :\to \mathbb{C}$ be a continuous function which is analytic in Ω and does not assume the value 0 on $\partial\Omega$. Furthermore assume that all zeros of f in Ω are simple. Then

$$\mathrm{d}(f, \Omega, 0) = n,$$

where n is the number of solutions of the equation

$$f(z) = 0$$

which lie inside Ω.

Solution: A zero z_0 of an analytic function is a zero multiplicity m (simple if $m = 1$) if we can write

$$f(z) = (z - z_0)^m g(z),$$

where g is an analytic function such that $g(z_0) \neq 0$. Thus z_0 is a simple zero of f if $f(z_0) = 0,\ f'(z_0) \neq 0$. let us identify $\mathbb{C}$ with $\mathbb{R}^2$ and write

$$f(z) = U(x, y) + iV(x, y),$$

with U the real part and V the imaginary part. It follows from complex analysis that

$$f'(z) = U_x(x, y) + iV_x(x, y),$$

and hence $f'(z) \neq 0$ if and only if $U_x^2(x, y) + V_x^2(x, y) \neq 0$. On the other hand, if we consider f as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, then

$$\mathrm{det} f'(x, y) = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = U_x V_y - V_x U_y = U_x^2 + V_x^2,$$

where we have used the Cauchy Riemann equations. It therefore follows that 0 is a regular value for f and we may use mthe degree formula

$$\mathrm{d}(f, \Omega, 0) = \sum_{z \in f^{-1}(0)} \mathrm{sgn}\ \mathrm{det} f'(x, y),$$

which by the above equals

$$\sum_{i=1}^{n} 1 = n,$$

where n is the number of zeros of f.