1. Exercises-Contraction maps-Solutions

1.1. **Exercise.** It may be the case that $T : \mathbb{M} \to \mathbb{M}$ is not a contraction on the whole space $\mathbb{M}$, but rather a contraction on some neighborhood of a given point. In this case we have the following result:

Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let

$$B = \{x \in \mathbb{M} : \mathrm{d}(z, x) < \epsilon\},$$

where $z \in \mathbb{M}$ and $\epsilon > 0$ is a positive number and let $T : \overline{B} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T(y), T(x)) \leq k\mathrm{d}(x, y), \ \forall \ x, y \in B,$$

with contraction constant $k < 1$. Furthermore assume that

$$\mathrm{d}(z, T(z)) < \epsilon(1 - k).$$

Then T has a unique fixed point $x \in B$.

Solution: That there is only one fixed point follows immediately.

We note that for $x \in \overline{B}$ it follows that

$$\mathrm{d}(z, T(x)) \leq \mathrm{d}(z, T(z)) + \mathrm{d}(T(z), T(x)) \leq k\mathrm{d}(x, z) < \epsilon(1 - k) + k\epsilon \leq \epsilon,$$

hence

$$T : \overline{B} \to B.$$

On the other hand, $\overline{B}$ is a complete metric space with respect to the induced metric. Hence T has a unique fixed point in B.

1.2. **Exercise.** Let $(\mathbb{M}, \mathrm{d})$ be a metric space and let $T : \mathbb{M} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \ \forall x, y \in \mathbb{M}, \ x \neq y.$$

Further assume that

$$T : \mathbb{M} \to K,$$

where K is a compact subset of $\mathbb{M}$. Then T has a unique fixed point in $\mathbb{M}$.

Solution 1:(Earnshaw) We note that

$$T : K \to K,$$

and, for $n = 0, 1, 2, \cdots$

$$T^{n+1}(K) \subset T^n(K).$$

Since K is compact, it follows that

$$A := \cap_{n=1}^{\infty} T^n(K) \neq \emptyset$$

and is compact.

The function

$$\mathrm{d} : A \times A \to [0, \infty)$$

is a continuous function, hence will assume its maximum at a point $(x', y') \in A \times A$. But, if $x' \neq y'$, there will exist points $x, y \in A$ such that $T(x) = x'$, $T(y) = y'$. And so

$$\mathrm{d}(x', y') < \mathrm{d}(x, y),$$

contradicting that $\mathrm{d} : A \times A \to [0, \infty)$ assumes its maximum at (x', y'). Thus the set A is a singleton, say $A = \{x\}$. It follows that $T(x) = x$.

Solution 2:(Krtolica) Consider the function

$$f(x) = \mathrm{d}(x, T(x)) : K \to [0, \infty).$$

This function is continuous and bounded below. Since K is compact this function will assume its minimum at a point $y \in K$. This minimum must be zero, for otherwise

$$\mathrm{d}(T(y), T^2(y)) < \mathrm{d}(y, T(y)),$$

a contradiction. On the other hand $\mathrm{d}(y, T(y)) = 0$, implies that

$$T(y) = y.$$

Remark: The ideas used in Krtolica's proof may also be used to give a different proof of the contraction mapping principle. Which proceeds as follows (recall the hypotheses of the contraction mapping principle!):

(1) It follows (by induction) that for any $x \in \mathbb{M}$ and any natural number m

$$\mathrm{d}(T^{m+1}(x), T^m(x)) \leq k^m \mathrm{d}(T(x), x).$$

(2) Let

$$\alpha = \inf_{x \in \mathbb{M}} \mathrm{d}(T(x), x).$$

Then, if $\alpha > 0$, there exists $x \in \mathbb{M}$ such that

$$\mathrm{d}(T(x), x) < \frac{3}{2}\alpha$$

and hence for any m

$$\mathrm{d}(T^{m+1}(x), T^m(x)) \leq k^m \frac{3}{2}\alpha.$$

On the other hand,

$$\alpha \leq \mathrm{d}(T(T^m(x)), T^m(x)) = \mathrm{d}(T^{m+1}(x), T^m(x))$$

and thus

$$\alpha \leq k^m \frac{3}{2}\alpha,$$

for any $m \geq 1$, which is impossible, since $k < 1$.

(3) We choose a sequence $\{x_n\}$ (a minimizing sequence) such that

$$\lim_{n \to \infty} \mathrm{d}(T(x_n), x_n)) = \alpha = 0.$$

For any m, n the triangle inequality implies that

$$\mathrm{d}(x_n, x_m) \leq \mathrm{d}(T(x_n), x_n)) + \mathrm{d}(T(x_m), x_m)) + \mathrm{d}(T(x_n), T(x_m)).$$

And hence

$$(1 - k)\mathrm{d}(x_n, x_m) \leq \mathrm{d}(T(x_n), x_n)) + \mathrm{d}(T(x_m), x_m)).$$

Which implies that $\{x_n\}$ is a Cauchy sequence and hence has a limit x in $\mathbb{M}$. One now concludes that

$$\mathrm{d}(T(x), x) = 0$$

and thus x is a fixed point of T.

1.3. **Exercise.** In some applications it is the case that the mapping T is a Lipschitz mapping which is not necessarily a contraction, whereas some power of T is a contraction mapping.

Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let $T : \mathbb{M} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T^m(x), T^m(y)) \leq k\mathrm{d}(x, y),\ \forall x, y \in \mathbb{M},$$

for some $m \geq 1$, where $0 \leq k < 1$ is a constant. Then T has a unique fixed point in $\mathbb{M}$.

Solution: It follows from the contraction mapping principle that T^m has a unique fixed point $z \in \mathbb{M}$. Thus

$$z = T^m(z)$$

implies that

$$T(z) = TT^m(z) = T^m(T(z)).$$

Thus $T(z)$ is a fixed point of T^m and hence by uniqueness of such fixed points

$$z = T(z).$$

1.4. **Exercise.** It is often the case in applications that a contraction mapping depends upon other variables (parameters) also. If this dependence is continuous, then the fixed point will depend continuously upon the parameters, as well. This is the content of the following result.

Let (Λ, ρ) be a metric space and $(\mathbb{M}, \mathrm{d})$ a complete metric space and let

$$T : \Lambda \times \mathbb{M} \to \mathbb{M}$$

be a family of contraction mappings with uniform contraction constant k, i.e.,

$$\mathrm{d}\left(T(\lambda, x), T(\lambda, y)\right) \leq k\mathrm{d}(x, y),\ \forall \lambda \in \Lambda,\ \forall x, y \in \mathbb{M}.$$

Further more assume that for each $x \in \mathbb{M}$ the mapping

$$\lambda \mapsto T(\lambda, x)$$

is a continuous mapping from Λ to $\mathbb{M}$. Then for each $\lambda \in \Lambda$, $T(\lambda, \cdot)$ has a unique fixed point $x(\lambda) \in \mathbb{M}$, and the mapping

$$\lambda \mapsto x(\lambda),$$

is a continuous mapping from Λ to $\mathbb{M}$.

Solution: For $\lambda_1, \lambda_2 \in \Lambda$ we have

$$\begin{aligned}
\mathrm{d}\left(x(\lambda_1), x(\lambda_2)\right) &= \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_2))\right) \\
&\leq \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right) \\
&\quad +\mathrm{d}\left(T(\lambda_2, x(\lambda_1)), T(\lambda_2, x(\lambda_2))\right) \\
&\leq \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right) \\
&\quad +k\mathrm{d}\left(x(\lambda_1)), x(\lambda_2)\right).
\end{aligned}$$

Therefore

$$(1-k)\mathrm{d}\left(x(\lambda_1), x(\lambda_2)\right) \leq \mathrm{d}\left(T(\lambda_1, x(\lambda_1)), T(\lambda_2, x(\lambda_1))\right).$$

The result thus follows from the continuity of T with respect to λ for each fixed x.

1.5. **Exercise.** In this section we shall assume that $\mathbb{M}$ is a Banach space with norm $\|\cdot\|$, which also a Hilbert space, i.e, that $\mathbb{M}$ is an inner product space (over the field of complex numbers) with inner product $(\cdot,\cdot)$, related to the norm by

$$\|u\|^2 = (u,u),\ \forall u \in \mathbb{M}.$$

We call a mapping

$$T : \mathbb{M} \to \mathbb{M},$$

a monotone mapping provided that

$$\mathrm{Re}(T(u) - T(v), u - v) \geq 0,\ \forall u, v \in \mathbb{M},$$

where Rec denotes the real part of a complex number c.

The following result gives the existence of unique fixed points of monotone Lipschitz mappings, without the assumption that they be contraction mappings.

Let $\mathbb{M}$ be a Hilbert space and let

$$T : \mathbb{M} \to \mathbb{M},$$

be a monotone mapping such that for some constant $\beta > 0$

$$\|T(u) - T(v)\| \leq \beta\|u - v\|,\ \forall u, v \in \mathbb{M}.$$

Then for any $w \in \mathbb{M}$, the equation

$$u + T(u) = w \tag{1}$$

has a unique solution u.

Solution:

If $\beta < 1$, then the mapping

$$u \mapsto w - T(u),$$

is a contraction mapping and the result follows from the contraction mapping principle. Next, consider the case that $\beta \geq 1$. We note that for $\lambda \neq 0$, u is a solution of

$$u = (1-\lambda)u - \lambda T(u) + \lambda w, \tag{2}$$

if, and only if, u solves (1). Consider the mapping

$$T_\lambda(u) = (1-\lambda)u - \lambda T(u) + \lambda w.$$

It follows that

$$T_\lambda(u) - T_\lambda(v) = (1-\lambda)(u-v) - \lambda(T(u) - T(v)).$$

Using properties of the inner product, we obtain

$$\begin{aligned}\|T_\lambda(u) - T_\lambda(v)\|^2 \leq\ & \lambda^2\beta^2\|u-v\|^2 \\ & -2\mathrm{Re}\lambda(1-\lambda)(T(u)-T(v), u-v) \\ & +(1-\lambda)^2\|u-v\|^2.\end{aligned}$$

Therefore, if $0 < \lambda < 1$, the monotonicity of T implies that

$$\|T_\lambda(u) - T_\lambda(v)\|^2 \leq (\lambda^2\beta^2 + (1-\lambda)^2)\|u-v\|^2.$$

Choosing

$$\lambda = \frac{1}{\beta^2+1},$$

We obtain that T_λ satisfies a Lipschitz condition with Lipschitz constant k given by

$$k^2 = \frac{\beta^2}{\beta^2 + 1},$$

hence is a contraction mapping. The result thus follows by an application of the contraction mapping principle.