1. Exercises-Contraction maps

1.1. **Exercise.** It may be the case that $T : \mathbb{M} \to \mathbb{M}$ is not a contraction on the whole space $\mathbb{M}$, but rather a contraction on some neighborhood of a given point. In this case we have the following result:

Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let

$$B = \{x \in \mathbb{M} : \mathrm{d}(z, x) < \epsilon\},$$

where $z \in \mathbb{M}$ and $\epsilon > 0$ is a positive number and let $T : \overline{B} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T(y), T(x)) \leq k\mathrm{d}(x, y),\ \forall\ x, y \in B,$$

with contraction constant $k < 1$. Furthermore assume that

$$\mathrm{d}(z, T(z)) < \epsilon(1 - k).$$

Then T has a unique fixed point $x \in B$.

1.2. **Exercise.** Let $(\mathbb{M}, \mathrm{d})$ be a metric space and let $T : \mathbb{M} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y),\ \forall x, y \in \mathbb{M},\ x \neq y.$$

Further assume that

$$T : \mathbb{M} \to K,$$

where K is a compact subset of $\mathbb{M}$. Then T has a unique fixed point in $\mathbb{M}$.

1.3. **Exercise.** In some applications it is the case that the mapping T is a Lipschitz mapping which is not necessarily a contraction, whereas some power of T is a contraction mapping.

Let $(\mathbb{M}, \mathrm{d})$ be a complete metric space and let $T : \mathbb{M} \to \mathbb{M}$ be a mapping such that

$$\mathrm{d}(T^m(x), T^m(y)) \leq k\mathrm{d}(x, y),\ \forall x, y \in \mathbb{M},$$

for some $m \geq 1$, where $0 \leq k < 1$ is a constant. Then T has a unique fixed point in $\mathbb{M}$.

1.4. **Exercise.** It is often the case in applications that a contraction mapping depends upon other variables (parameters) also. If this dependence is continuous, then the fixed point will depend continuously upon the parameters, as well. This is the content of the following result.

Let (Λ, ρ) be a metric space and $(\mathbb{M}, \mathrm{d})$ a complete metric space and let

$$T : \Lambda \times \mathbb{M} \to \mathbb{M}$$

be a family of contraction mappings with uniform contraction constant k, i.e.,

$$\mathrm{d}\left(T(\lambda, x), T(\lambda, y)\right) \leq k\mathrm{d}(x, y),\ \forall \lambda \in \Lambda,\ \forall x, y \in \mathbb{M}.$$

Further more assume that for each $x \in \mathbb{M}$ the mapping

$$\lambda \mapsto T(\lambda, x)$$

is a continuous mapping from Λ to $\mathbb{M}$. Then for each $\lambda \in \Lambda$, $T(\lambda, \cdot)$ has a unique fixed point $x(\lambda) \in \mathbb{M}$, and the mapping

$$\lambda \mapsto x(\lambda),$$

is a continuous mapping from Λ to $\mathbb{M}$.

1.5. **Exercise.** In this section we shall assume that $\mathbb{M}$ is a Banach space with norm $\|\cdot\|$, which also a Hilbert space, i.e, that $\mathbb{M}$ is an inner product space (over the field of complex numbers) with inner product $(\cdot,\cdot)$, related to the norm by

$$\|u\|^2 = (u,u),\ \forall u \in \mathbb{M}.$$

We call a mapping

$$T : \mathbb{M} \to \mathbb{M},$$

a monotone mapping provided that

$$\mathrm{Re}(T(u) - T(v), u - v) \geq 0,\ \forall u, v \in \mathbb{M},$$

where Rec denotes the real part of a complex number c.

The following result gives the existence of unique fixed points of monotone Lipschitz mappings, without the assumption that they be contraction mappings.

Let $\mathbb{M}$ be a Hilbert space and let

$$T : \mathbb{M} \to \mathbb{M},$$

be a monotone mapping such that for some constant $\beta > 0$

$$\|T(u) - T(v)\| \leq \beta\|u - v\|,\ \forall u, v \in \mathbb{M}.$$

Then for any $w \in \mathbb{M}$, the equation

$$u + T(u) = w \tag{1}$$

has a unique solution u.

1.5.1. *Outline of proof.* If $\beta < 1$, then the mapping

$$u \mapsto w - T(u),$$

is a contraction mapping and the result follows from the contraction mapping principle. Next, consider the case that $\beta \geq 1$. We note that for $\lambda \neq 0$, u is a solution of

$$u = (1-\lambda)u - \lambda T(u) + \lambda w, \tag{2}$$

if, and only if, u solves (1). Consider the mapping

$$T_\lambda(u) = (1-\lambda)u - \lambda T(u) + \lambda w.$$