SAMPLE EXAM II, MATH 2270-1

1) Compute the determinant using the co-factor expansion along a row or a column of your choice.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

2) Find an inverse of the matrix A by performing the row reduction to $[A|I]$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

3) Find a basis of the null space and the column space of the matrix A using the given row echelon form.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4) Which of the following three subsets of $\mathbb{R}^2$ is a subspace? Why or why not? (Here $\mathbb{Z}$ denotes the set of integers.)

(1) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{Z} \right\}$.

(2) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 + x_2 = 0 \right\}$.

(3) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 \cdot x_2 = 0 \right\}$.

5) Suppose $v_1, \ldots, v_m$ are vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that the vectors $T(v_1) \ldots, T(v_m)$ are linearly independent. Prove that $v_1, \ldots, v_m$ are linearly independent.

Solutions:

1) Expanding along the first column:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 6 - 5 + 1 = 2$$

2) Answer:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}.$$

3) A basis of the column is the first two columns of A since the non-zero pivots in the reduced form appear in those two columns. Null space is the space of solutions to the homogeneous system. From the echelon form we see that $x_3 = s$ and $x_4 = t$ are free variables, $x_2 = -2s - 3t$ and $x_1 = s + 2t$. Thus any solution in vector form is

$$\begin{bmatrix} s+2t \\ -2s-3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

so a basis is the two column vectors on the right hand side.

4)

(1) No, since $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V$ but $\frac{1}{2}v \notin V$.

(2) Yes, it is a kernel of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$. (Which one?)

(3) No, since $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$ and $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$ but $v + u \notin V$.

5) Want to show that $x_1 = \ldots = x_m = 0$ is the unique solution of the equation $x_1 v_1 + \ldots + x_m v_m = 0$. Apply T to this equation:

$$T(x_1 v_1 + \ldots + x_m v_m) = T(0).$$

Since T is a linear trasformation, $T(x_1 v_1 + \ldots + x_m v_m) = x_1 T(v_1) + \ldots + x_m T(v_m)$ and $T(0) = 0$. Substituting these, we have

$$x_1 T(v_1) + \ldots + x_m T(v_m) = 0.$$

Since $T(v_1), \ldots T(v_m)$ are linearly independent it follows that $x_1 = \ldots = x_m = 0$.