1) Consider the surface $x^2 + z = 2$ and the parametrized curve 

$$ \sigma(t) = (1, t^3, t). $$

a) Find the point $P$ where the curve intersects the surface.
b) Write a parametric equation of the line tangent to $\sigma(t)$ at the point $P$.
c) Write an equation of the tangent plane to the surface at the point $P$.

Answers: a) $(1, 1, 1)$ (for $t = 1$); b) $(1, 1, 1) + t(0, 3, 1)$; c) $2x + z = 3$.

2) a) Sketch the region of integration for the integral

$$ \int_{0}^{1} \int_{\sqrt{y}}^{1} \sqrt{y} \cos(x^4) \, dx \, dy. $$

b) Compute the integral (Hint: reverse the order of integration).

Answer: $\frac{\sin 1}{6}$.

3) Find the minimum and the maximum value of the function $f(x, y) = 3x + 4y$ along the circle $x^2 + y^2 = 1$.

Answer: Maximum is $f(3/5, 4/5) = 5$ and minimum $f(-3/5, -4/5) = -5$.

4) Find and analyze the critical points of the function

$$ f(x, y) = 2x^4 - x^2 + 3y^2. $$

Solution: $f_x = 8x^3 - 2x = 2x(4x^2 - 1)$, $f_y = 6y^2$. Thus $f_x = 0$ on vertical lines $x = 0, 1/2$ and $-1/2$, and $f_y = 0$ on the horizontal line $y = 0$. Thus $f_x = 0$ AND $f_y = 0$ for three points $(0, 0), (1/2, 0)$ and $(-1/2, 0)$. These are the critical points. Since

$$ \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 24x^2 - 2 & 0 \\ 0 & 6 \end{pmatrix} $$

second derivative test says that $(0, 0)$ is a saddle point, while $(\pm 1/2, 0)$ are local minima.

5) Evaluate the line integral by finding a potential function.

$$ \int_{(1,1)}^{(2,2)} (3x^2 + 2xy^2) \, dx + (3y^2 + 2x^2y) \, dy. $$

Solution: $f(x, y) = x^3 + x^2y^2 + y^3$.

6) Consider the vector field $\nabla = xi + yj$. Calculate the flux of $\nabla$ across the boundary of the triangle with vertices $(0, 0), (1, 0)$ and $(0, 1)$ in two ways:

a) Directly.
b) Using the Divergence theorem.

Solution: Part a). The flux across $C$ is given by

$$\int_C \nabla \cdot \mathbf{n} \, ds$$

where $\mathbf{n}$ is the vector of length one and perpendicular to the curve $C$. The curve $C$ is a triangle, so the vector $\mathbf{n}$ is constant along each edge. Consider first the edge $C_1$ connecting $(0, 1)$ and $(1, 0)$. This edge lies on the line $x + y = 1$ i.e. on the graph of $y = 1 - x$, with $0 \leq x \leq 1$. Along this edge $\mathbf{n} = \frac{i + j}{\sqrt{2}}$. Thus

$$(\nabla \cdot \mathbf{n}) = \frac{x + y}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$  

Since $ds = \sqrt{1 + (f'(x))^2} \, dx$ and $f(x) = 1 - x$ for the edge in question, $ds = \sqrt{2} \, dx$. Putting everything together

$$\int_{C_1} \nabla \cdot \mathbf{n} \, ds = \int_0^1 \frac{1}{\sqrt{2}} \, \sqrt{2} \, dx = 1.$$  

The flux along the other two edges is 0, since $\nabla \cdot \mathbf{n} = 0$ there, so the total flux is 1. Part b) is easy, since $\text{div} \mathbf{V}(x, y) = 2$, so the double integral of the triangle is 1.

7) Consider the vector field $\mathbf{V}(x, y) = -yi + xj$. Let $\mathbf{V}_\varphi$ be the vector field obtained by rotating $\mathbf{V}$ counter-clockwise for the angle $\varphi$. Find the flux of $\mathbf{V}_\varphi$ across the circle $x^2 + y^2 = 1$ two ways:

a) Directly.

b) Using the divergence theorem.

Solution:

$$\mathbf{V}_\varphi(x, y) = (-y \cos \varphi - x \sin \varphi)i + (-y \sin \varphi + x \cos \varphi)j.$$  

The flux across $C$ is given by

$$\int_C \mathbf{V}_\varphi \cdot \mathbf{n} \, ds$$

where $\mathbf{n}$ is the vector of length one and perpendicular to the curve $C$. (Of course $\mathbf{n}$ depends on the point on $C$.) For this particular curve, unit circle, $\mathbf{n} = xi + yj$. Since $x^2 + y^2 = 1$ on $C$,

$$\mathbf{V}_\varphi \cdot \mathbf{n} = -x^2 \sin \varphi - y^2 \sin \varphi = \sin \varphi.$$  

Thus

$$\int_C \mathbf{V}_\varphi \cdot \mathbf{n} \, ds = \int_C -\sin \varphi \, ds = -\sin \varphi \int_C ds = -2\pi \sin \varphi$$

since the length of $C$ is $2\pi$. This is “direct” calculation of the flux. Using the divergence theorem, let $S$ be the interior of the unit circle,

$$\int \int_S \text{div} \mathbf{V}_\varphi \, dxdy = \int \int_S -2 \sin \varphi \, dxdy = -2\sin \varphi \int \int_S dxdy = -2\pi \sin \varphi$$

since the area of $S$ is $\pi$. 
7) Consider the vector field \( \mathbf{V}(x, y) = -yi + xj \). Let \( \mathbf{V}_\varphi \) be the vector field obtained by rotating \( \mathbf{V} \) counter-clockwise for the angle \( \varphi \). Find the work done of \( \mathbf{V}_\varphi \) along the circle \( x^2 + y^2 = 1 \) in the counterclockwise direction in two ways:
   a) Directly.
   b) Using Green’s theorem.
   The work done along \( C \) is given by
   \[
   \int_C (-y \cos \varphi - x \sin \varphi) \, dx + (-y \sin \varphi + x \cos \varphi) \, dy.
   \]
   In order to compute this line integral, we need to parameterize the curve \( C \). This is done by \( x = \cos t \) and \( y = \sin t \) where \( 0 \leq t \leq 2\pi \). Since \( dx = -\sin t \, dt \) and \( dy = \cos t \, dt \), the integral is
   \[
   \int_0^{2\pi} (-\sin t \cos \varphi - \cos t \sin \varphi)(-\sin t) \, dt + (-\sin t \sin \varphi + \cos t \cos \varphi)(\cos t) \, dt = \\
   \int_0^{2\pi} (\sin^2 t \cos \varphi + \cos^2 t \cos \varphi) \, dt = \int_0^{2\pi} \cos \varphi \, dt = 2\pi \cos \varphi.
   \]
   Using Green’s theorem,
   \[
   \int \int_S (\cos \varphi, \sin \varphi) \cdot (-y \cos \varphi - x \sin \varphi) \, dxdy = \int \int_S 2 \cos \varphi \, dxdy = 2\pi \cos \varphi.
   \]

8) Let \( C \) be the closed curve obtained by going from \((0, 0)\) to \((1, -1)\) along \( y = -x^2 \), then going to \((1, 1)\) along \( x = 1 \), and then coming back to \((0, 0)\) along \( y = x^2 \). Calculate \( \int_C x \, dy \) two ways:
   a) Directly.
   b) Using Green’s theorem.