MATH 5310 FOURTH EXAM PROBLEMS

1) The octahedral group O is the group of rotations of a cube. Let S be the set of vertices of the cube. Then O acts transitively on the S. Determine the stabilizer in O of a vertex. Use this to compute the order of O.

2) Let X be the set of 2-dimensional subspaces of $\mathbb{F}_p^n$, where $n \geq 2$.

(1) Show that $GL_n(\mathbb{F}_p)$ acts transitively on X.

(2) Compute the stabilizer S in $GL_n(\mathbb{F}_p)$ of the 2-dimensional subspace

$$U = \{(x_1, x_2, 0, \ldots, 0) \in \mathbb{F}_p^n \mid x_1, x_2 \in \mathbb{F}_p\}.$$

(3) Compute the order of X.

Note: the order of $GL_n(\mathbb{F}_p)$ is $(p^n - 1)(p^n - p) \cdot \cdots \cdot (p^n - p^{n-1})$.

3) Let p be an odd prime. Compute the centralizers in $GL_2(\mathbb{F}_p)$, and their orders, of the following two elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hint: The order of the centralizer of the second element depends on whether -1 is a square in $\mathbb{F}_p$ or not.

4) Prove the fixed point theorem: Let p a prime and G a group of order p^e. If G acts on a set X of order prime to p then G fixes an element in X.

5) Determine the class equation of the group of matrices with coefficients in $\mathbb{F}_p$:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

6) The class equation of a group G is 20=1+4+5+5+5. Show that G has groups of order 4 and 5, the group of order 5 is normal, while the groups of order 4 are not.

Solutions:

1) Let c be the center of the cube. Fix a vertex x. There are three rotations that fix the vertex x: rotations about the line trough c and x and the angle 0, 120 and 240 degrees. Hence the order of the stabilizer O_x of x in O is 3. The group O acts transitively on S, hence $|O| = |S| \cdot |T_x| = 8 \cdot 3 = 24$.

2) (1) Let U be the 2-dimensional subspace as in (2), so it is spanned by e_1 and e_2 of the standard basis of $\mathbb{F}_p^n$. Let V be any 2-dimensional subspace. Let v_1 and v_2 be a basis of V and complete it to a basis of $\mathbb{F}_p^n$. Let $g \in GL_n(\mathbb{F}_p)$ whose columns are v_1, v_2 etc. Then $g(U) = V$, hence $GL_n(\mathbb{F}_p)$ acts transitively on X.

3) The centralizer of the first element is the group of diagonal matrices, which has $(p-1)^2$ elements. The centralizer of the second element is the group of invertible matrices of the type

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The determinant is $a^2 + b^2$ so, in order to count the number of invertible matrices, we need to exclude those such that $a^2 + b^2 = 0$. If -1 is not a square then the only solution of this equation is $a = b = 0$. Thus we have $p^2 - 1$ invertible matrices in this case. Otherwise, if -1 is a square, i.e. we have an element in $\mathbb{F}_p$ such that $i^2 = -1$, then we can factor $a^2 + b^2 = (a - ib)(a + bi)$. Thus $a^2 + b^2 \neq 0$ as long as we avoid the union of two lines, $a - ib = 0$ and $a + bi$, which intersect in $a = b = 0$, for a total of $2p - 1$ points. Thus the centralizer has $p^2 - (2p - 1) = p^2 - 2p + 1 = (p-1)^2$ elements.

Here is another way to do the case -1 a square. The eigenvalues of the second matrix are $\pm\sqrt{-1}$ hence, if -1 is a square, it can be diagonalized with i and $-i$ on the diagonal. Now the centralizer is the group of diagonal matrices, by the same computation as for the first matrix.

4) Let $x \in X$ and let O_x be the G-orbit of x. Then $|O_x| = |G|/|G_x|$ where G_x is the stabilizer of x in G. The point x is a fixed if $|O_x| = 1$. Otherwise, the order of the orbit is divisible by p since $|G| = p^e$. Hence, if there is no fixed point, then the order of any orbit is divisible by p and the order of X is divisible by p, by the class equation, a contradiction.

5) The key is to compute the centralizers of elements. If the element is central i.e. $x = y = 0$ then the centralizer is the whole group, otherwise it is a subgroup of p^2 elements. Thus a conjugacy class has either 1 or p elements. There are p central elements and the rest of the group is a union of $p^2 - 1$ conjugacy classes, with p elements in each, since $p^3 = p \cdot 1 + (p^2 - 1) \cdot p$.

6) Let x be in one of the three conjugacy classes of order 5. Then the stabilizer G_x of x has order 4. (In particular, since $x \in G_x$, the order of x is 2 or 4.) A group of order 4 is too small to be a union of G-conjugacy classes, hence it is not normal. If x is in the conjugacy class of order 4 then G_x has order 5. This group cannot contain elements of order 2 or 4 hence G_x is the union of two conjugacy classes of the total order $1 + 4 = 5$. Hence G_x is normal.