MATH 5310 THIRD EXAM PROBLEMS

1) Let $F[x]$ be the space of polynomials over a field F. Let $p(x)$ be a polynomial of degree n. Let I be the set of all polynomials divisible by $p(x)$. Show that I is a subspace of $F[x]$. The quotient space $V = F[x]/I$ is the set of congruence classes of polynomials modulo $p(x)$ i.e. $f(x) \equiv g(x) \pmod{p(x)}$ if $p(x)$ divides the difference $f(x) - g(x)$. Prove that $1, x, \ldots, x^{n-1}$ is a basis of V.

2) Let V be the space of polynomials (over any field) modulo $x^{100} - 1$. Let $T : V \to V$ be the linear transformation given by multiplication by x . Find the characteristic polynomial of this transformation, using the basis $1, x, \ldots, x^{99}$.

3) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation given by a 120 degrees rotation in the counterclockwise direction about the ray through the vector $(1, 1, 1)$. Write down the matrix of T in the standard basis of $\mathbb{R}^3$.

4) Let T be a linear transformation on a complex vector space V such that $T^n = 1$. Show that, for every $v \in V$, $v + T(v) + \ldots + T^{n-1}(v)$ is an eigenvector with eigenvalue 1. If n is even, modify this a bit to construct an eigenvector with -1. If n is odd, can we have an eigenvector with eigenvalue -1?

5) Let V be a vector space over a field F and $v_1, \ldots, v_n$ eigenvectors with pairwise different eigenvalues $\lambda_1, \ldots, \lambda_n$. Use the induction on n to prove that $v_1, \ldots, v_n$ are linearly independent.

6) Let T be a linear transformation on a complex vector space V such that $T^2 = -1$. Show that any vector $v \in V$ can be written as a sum of two eigenvectors with eigenvalues i and $-i$. Then use the previous problem to show that V is isomorphic to a direct sum of two eigenspaces for T.

7) Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. Find a 2×2 matrix P such that $P^{-1}AP$ is diagonal. Then compute A^{100}.

8) Find invariant subspaces for $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$