MATH 5310 THIRD EXAM PROBLEMS

1) Let p be an odd prime. Compute the centralizers in $GL_2(\mathbb{F}_p)$, and their orders, of the following two elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hint: The order of the centralizer of the second element depends on whether -1 is a square in $\mathbb{F}_p$ or not.

2) Prove the fixed point theorem: Let p a prime and G a group of order p^e. If G acts on a set X of order prime to p then G fixes an element in X.

3) Let X be the set of 2-dimensional subspaces of $\mathbb{F}_p^n$, where $n \geq 2$.

(1) Compute the order of X.

(2) Compute the stabilizer S in $GL_n(\mathbb{F}_p)$ of the 2-dimensional subspace

$$U = \{(x_1, x_2, 0, \ldots, 0) \in \mathbb{F}_p^n \mid x_1, x_2 \in \mathbb{F}_p\}.$$

(3) Compute the order of S.

(4) Show that $GL_n(\mathbb{F}_p)$ acts transitively on X.

Note: the order of $GL_n(\mathbb{F}_p)$ is $(p^n - 1)(p^n - p) \cdot \cdots \cdot (p^n - p^{n-1})$.

4) Describe the class equation of the tetrahedral group T. Use the equation to show that there is only one subgroup of order 4 in T.

5) The class equation of a group G is 20=1+4+5+5+5. Show that G has groups of order 4 and 5, the group of order 5 is normal, while the groups of order 4 are not. Do not use the Sylow theorems.