1) Use the Eiseinstein Criterion to prove that \( x^6 + x^3 + 1 \) is irreducible.

Solution: Replace \( x \) by \( x + 1 \), then apply the criterion with \( p = 3 \).

2) Let \( \varphi : \mathbb{Z}[x] \to \mathbb{C} \) be the map defined by \( f(x) \mapsto f(1 + i) \). Let \( I \) be the kernel of \( \varphi \). Prove that \( I \) is principal, i.e. find a generator \( g(x) \) and prove that any element in \( I \) is a multiple of \( g(x) \).

Solution: \( g(x) = (x - (1 + i))(x - (1 - i)) = x^2 - 2 + 2 \). Let \( f(x) \in \mathbb{Z}[x] \). Then, since \( g(x) \) is monic,

\[
f(x) = h(x)g(x) + ax + b
\]

for some \( h(x) \in \mathbb{Z}[x] \) and \( a, b \in \mathbb{Z} \). If \( f(x) \in I \) then, after substituting \( x = 1 + i \) in the above equation, we get \( 0 = a(1 + i) + b \). Since \( 1 + i \) and \( 1 \) are linearly independent over \( \mathbb{Q} \), \( a = b = 0 \). Thus \( f(x) \) is a multiple of \( g(x) \).

3) Prove that the ring \( \mathbb{Z}[\sqrt{-2}] \) is euclidean with respect to the norm \( N(x + y\sqrt{-2}) = x^2 + 2y^2 \), i.e. for every \( \alpha, \beta \in \mathbb{Z}[\sqrt{-2}] \), with \( \beta \neq 0 \), show that there exists \( \gamma, \delta \in \mathbb{Z}[\sqrt{-2}] \), such that \( \alpha = \gamma \beta + \delta \), and \( N(\delta) < N(\beta) \). Do this for \( \alpha = 4 + 2\sqrt{-2} \) and \( \beta = 1 + \sqrt{-2} \).

Solution: \( \gamma \) is an element in \( \mathbb{Z}[\sqrt{-2}] \), closest to \( \alpha/\beta \). Let \( \eta = \alpha/\beta - \gamma \). Then \( \eta \) is in the Voronoi polygon for the lattice \( \mathbb{Z}[\sqrt{-2}] \):

\[
\{ x + y\sqrt{-2} \mid |x|, |y| \leq 1/2 \}.
\]

One sees that the polygon is strictly contained in the unit circle. Hence \( N(\eta) < 1 \) and this is equivalent to \( N(\delta) < N(\beta) \).

4) Let \( R = \mathbb{Z}[\sqrt{-2}] \). Let \( p \) be a prime. When is the principal ideal \( (p) \subseteq R \) maximal? (Hint: use \( R \cong \mathbb{Z}[x]/(x^2 + 2) \).) Use this to determine primes \( p \) that can be written as \( p = x^2 + 2y^2 \). Using the quadratic reciprocity, the answer depends on what \( p \) modulo 8 is, as Gauss in german say would.

Solution: Since \( R \cong \mathbb{Z}[x]/(x^2 + 2) \),

\[
R/(p) \cong \mathbb{Z}[x]/(p, x^2 + 2) \cong \mathbb{F}_p[x]/(x^2 + 2).
\]

Thus \( (p) \) is maximal in \( R \) if and only if \( (x^2 + 2) \) is maximal in \( \mathbb{F}_p[x] \). Since ideals in \( \mathbb{F}_p[x] \) are principal, ideals containing \( (x^2 + 2) \) corresponds to divisors of \( x^2 + 2 \). Thus \( (p) \) is maximal if and only if \( -2 \) is not a square modulo \( p \). (The quadratic reciprocity says that \( -2 \) is a square modulo an odd prime \( p \) if and only if \( p \equiv 1, 3 \pmod{8} \).)

Now, if \( x^2 + 2y^2 = p \) then \( (x/y)^2 \equiv -2 \pmod{p} \), i.e. \( -2 \) is a square modulo \( p \). Conversely, if \( -2 \) is a square modulo \( p \), then there exists an ideal \( P \) such that \( R \supset P \supset (p) \). The norm of \( P \) is \( p \), since \( N(P) \) is a proper divisor of \( N(p) = p^2 \). Since \( P = (x + y\sqrt{-2}) \) by problem 3, \( p = N(P) = x^2 + 2y^2 \).