1) Use the Eiseinstein Criterion to prove that \( x^6 + x^3 + 1 \) is irreducible. (Hint: replace \( x \) by \( x + 1 \).)

2) Let \( \varphi : \mathbb{Z}[x] \to \mathbb{C} \) be the map defined by \( f(x) \mapsto f(1 + i) \). Let \( I \) be the kernel of \( \varphi \). Prove that \( I \) is principal, i.e. find a generator \( g(x) \) and prove that any element in \( I \) is a multiple of \( g(x) \). Hint: you will need to use Gauss’ Lemma for this.

3) Prove that the ring \( \mathbb{Z}[\sqrt{-2}] \) is euclidean with respect to the norm \( N(x + y\sqrt{-2}) = x^2 + 2y^2 \), i.e. for every \( \alpha, \beta \in \mathbb{Z}[\sqrt{-2}] \), with \( \beta \neq 0 \), show that there exists \( \gamma, \delta \in \mathbb{Z}[\sqrt{-2}] \), such that \( \alpha = \gamma \beta + \delta \), and \( N(\delta) < N(\beta) \). Do this for \( \alpha = 4 + 2\sqrt{-2} \) and \( \beta = 1 + \sqrt{-2} \).

4) Let \( R = \mathbb{Z}[\sqrt{-2}] \). Let \( p \) be a prime. When is the principal ideal \( (p) \subseteq R \) maximal? (Hint: use \( R \cong \mathbb{Z}[x]/(x^2 + 2) \) to understand \( R/(p) \).) Use this to determine primes \( p \) that can be written as \( p = x^2 + 2y^2 \). Using the quadratic reciprocity, the answer depends on what \( p \) modulo 8 is, as Gauss in german say would.

5) Let \( R \) be a ring such that any ideal is finitely generated. Let \( I_1 \subseteq I_2 \subseteq I_2 \subseteq \ldots \) be an infinite sequence of ideals in \( R \). Prove that exists an integer \( n \) such that \( I_n = I_{n+1} = \ldots \).

6) Find the quotient \( \mathbb{Z}^3/N \) (in the normal form) where \( N \) is a submodule generated by the columns of the matrix
\[
\begin{pmatrix}
3 & 1 & 2 & 4 \\
1 & 1 & 1 & 1 \\
2 & 3 & 6 & 5
\end{pmatrix}
\]