MATH 5320 - SAMPLE FIRST EXAM

1) Let I, J be ideals in a ring R. Show that $I + J = \{r \in R \mid r = x + y, x \in I, y \in J\}$ is an ideal. If $(n), (m)$ are two principal ideals in $\mathbb{Z}$, what is $(n) + (m)$?

Solution: The second part. Let d be the greatest common divisor of n and m. Since $(n) + (m)$ is the set of all integers $xn + ym$, and these elements are multiples of d, we have $(n) + (m) \subseteq (d)$. Since $xn + ym = d$ has a solution in x and y, the equality $(n) + (m) = (d)$ holds.

2) Let $\varphi : \mathbb{Z}[x] \to \mathbb{R}$ be the map defined by $f(x) \mapsto f(1/2)$. Let I be the kernel of φ. Prove that I is principal, i.e. find a generator $g(x)$ and prove that any element in I is a multiple of $g(x)$.

Solution: Let $g(x) = 2x - 1$. Since $g(x)$ is not monic, we cannot divide any $f(x) \in \mathbb{Z}[x]$. Assume that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in I$. Since $f(1/2) = 0$, after multiplying by 2^n we get

$$a_n + a_{n-1}2 + \ldots + a_0 2^n = 0.$$

It follows that a_n is even, so we can start dividing $f(x)$ by $2x - 1$ in $\mathbb{Z}[x]$:

$$f(x) - \frac{a_n}{2}(2x - 1) = h(x)$$

where the degree of h is smaller than the degree of f. Substitute $x = 1/2$ in the above equation. The left hand side is clearly zero, hence $h(1/2) = 0$. If we already know that $h(x)$ is a multiple of $2x - 1$ then the above equation implies that $f(x)$ is also a multiple of $2x - 1$. Thus the proof follows by the induction on the degree of polynomials in I. The induction base: if $f(x) \in I$ has degree 0, then $f(x) = 0$, clearly a multiple of $2x - 1$.

3) What are the maximal ideals in $\mathbb{R}[x]/(x^2 - 3x + 2)$? In $\mathbb{R}[x]/(x^2 + x + 1)$?

Solution: Maximal ideals in a quotient ring R/I correspond to maximal ideals in R containing I. Apply this to $R = \mathbb{R}[x]$. In this ring any ideal I is principal, $I = (f(x))$. Then $J = (g(x))$ contains I if and only if $f(x)$ is a multiple of $g(x)$. Furthermore, J is maximal if and only if $g(x)$ is an irreducible polynomial. Since $x^2 - 3x + 2 = (x - 1)(x - 2)$, the first ring has two maximal ideals generated by $x - 1$ and $x - 2$, respectively. On the other hand, $x^2 + x + 1$ is irreducible, so only the 0 ideal is maximal. (This ring is a field isomorphic to $\mathbb{C}$.)

4) Is the ring $\mathbb{Z}/10\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$? Is $\mathbb{Z}/8\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$?

Solution: The map $x \mapsto (x, x)$ defines a homomorphism from $\mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. The kernel is $\mathbb{Z}/10\mathbb{Z}$, yielding an isomorphism by the first isomorphism theorem. For the second part, writing a ring as a product is the same as finding non-trivial idempotents. But $\mathbb{Z}/8\mathbb{Z}$ has no idempotents except 0 and 1, by a simple check.

5) Determine the ring $\mathbb{Z}[x]/(x^2-1, 2x+1)$.

Solution: The idea is to divide x^2+1 by $2x+1$. This can be done provided we take an appropriate multiple of x^2+1:

$$2(x^2+1) - x(2x+1) = -x+2$$

Thus $-x+2$ is also in the ideal. Dividing once again, $2(-x+2)+(2x+1) = 5$ is also in the ideal. Hence

$$\mathbb{Z}[x]/(x^2+1, 2x+1) = \mathbb{Z}[x]/(x^2+1, 2x+1, 5) \cong \mathbb{F}_5[x]/(x^2+1, 2x+1)$$

where the last isomorphism is the correspondence of ideals in $\mathbb{F}_5[x]$ and ideals in $\mathbb{Z}[x]$ containing (5). Since $2 \cdot 3 = 1$ in $\mathbb{F}_5$, we can replace $2x-1$ by the monic $3(2x-1) = x-3$. Now, using the division, one checks that $x-3$ divides of x^2+1. Hence

$$\mathbb{F}_5[x]/(x^2+1, 2x+1) = \mathbb{F}_5[x]/(x^2+1, x-3) = \mathbb{F}_5[x]/(x-3) \cong \mathbb{F}_5$$

where the last isomorphism is given by evaluating polynomials at 3.