1) Let \( I, J \) be ideals in a ring \( R \). Show that \( I + J = \{ r \in R \mid r = x + y, x \in I, y \in J \} \) is an ideal. If \((n), (m)\) are two principal ideals in \( \mathbb{Z} \), what is \((n) + (m)\)?

   Solution: The second part. Let \( d \) be the greatest common divisor of \( n \) and \( m \). Since \((n) + (m)\) is the set of all integers \( xn + ym \), and these elements are multiples of \( d \), we have \((n) + (m) \subseteq (d)\). Since \( xn + ym = d \) has a solution in \( x \) and \( y \), the equality \((n) + (m) = (d)\) holds.

2) Let \( \varphi : \mathbb{Z}[x] \to \mathbb{R} \) be the map defined by \( f(x) \mapsto f(1/2) \). Let \( I \) be the kernel of \( \varphi \). Prove that \( I \) is principal, i.e. find a generator \( g(x) \) and prove that any element in \( I \) is a multiple of \( g(x) \).

   Solution: Let \( g(x) = 2x - 1 \). Since \( g(x) \) is not monic, we cannot divide any \( f(x) \in \mathbb{Z}[x] \). Assume that \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \in I \). Since \( f(1/2) = 0 \), after multiplying by \( 2^n \) we get
   \[
   a_n + a_{n-1}2 + \ldots + a_02^n = 0.
   \]
   It follows that \( a_n \) is even, so we can start dividing \( f(x) \) by \( 2x - 1 \) in \( \mathbb{Z}[x] \):
   \[
   f(x) - \frac{a_n}{2}(2x - 1) = h(x)
   \]
   where the degree of \( h \) is smaller than the degree of \( f \). Substitute \( x = 1/2 \) in the above equation. The left hand side is clearly zero, hence \( h(1/2) = 0 \). If we already know that \( h(x) \) is a multiple of \( 2x - 1 \) then the above equation implies that \( f(x) \) is also a multiple of \( 2x - 1 \). Thus the proof follows by the induction on the degree of polynomials in \( I \). The induction base: if \( f(x) \in I \) has degree 0, then \( f(x) \) is 0, clearly a multiple of \( 2x - 1 \).

3) What are the maximal ideals in \( \mathbb{R}[x]/(x^2 - 3x + 2) \)? In \( \mathbb{R}[x]/(x^2 + x + 1) \)?

   Solution: Maximal ideals in a quotient ring \( R/I \) correspond to maximal ideals in \( R \) containing \( I \). Apply this to \( R = \mathbb{R}[x] \). In this ring any ideal \( I \) is principal, \( I = (f(x)) \). Then \( J = (g(x)) \) contains \( I \) if and only if \( f(x) \) is a multiple of \( g(x) \). Furthermore, \( J \) is maximal if and only if \( g(x) \) is an irreducible polynomial. Since \( x^2 - 3x + 2 = (x - 1)(x - 2) \), the first ring has two maximal ideals generated by \( x - 1 \) and \( x - 2 \), respectively. On the other hand, \( x^2 + x + 1 \) is irreducible, so only the 0 ideal is maximal. (This ring is a field isomorphic to \( \mathbb{C} \).)

4) Is the ring \( \mathbb{Z}/10\mathbb{Z} \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \)? Is \( \mathbb{Z}/8\mathbb{Z} \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \)?

   Solution: The map \( x \mapsto (x, x) \) defines a homomorphism from \( \mathbb{Z} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \). The kernel is \( \mathbb{Z}/10\mathbb{Z} \), yielding an isomorphism by the first isomorphism theorem. For the second part, writing a ring as a product is the same as finding non-trivial idempotents. But \( \mathbb{Z}/8\mathbb{Z} \) has no idempotents except 0 and 1, by a simple check.
5) Determine the ring $\mathbb{Z}[x]/(x^2 - 1, 2x + 1)$.

Solution: The idea is to divide $x^2 + 1$ by $2x + 1$. This can be done provided we take an appropriate multiple of $x^2 + 1$:

$$2(x^2 + 1) - x(2x + 1) = -x + 2$$

Thus $-x + 2$ is also in the ideal. Dividing once again, $2(-x + 2) + (2x + 1) = 5$ is also in the ideal. Hence

$$\mathbb{Z}[x]/(x^2 + 1, 2x + 1) = \mathbb{Z}[x]/(x^2 + 1, 2x + 1, 5) \cong \mathbb{F}_5[x]/(x^2 + 1, 2x + 1)$$

where the last isomorphism is the correspondence of ideals in $\mathbb{F}_5[x]$ and ideals in $\mathbb{Z}[x]$ containing (5). Since $2 \cdot 3 = 1$ in $\mathbb{F}_5$, we can replace $2x - 1$ by the monic $3(2x - 1) = x - 3$. Now, using the division, one checks that $x - 3$ divides of $x^2 + 1$. Hence

$$\mathbb{F}_5[x]/(x^2 + 1, 2x + 1) = \mathbb{F}_5[x]/(x^2 + 1, x - 3) = \mathbb{F}_5[x]/(x - 3) \cong \mathbb{F}_5$$

where the last isomorphism is given by evaluating polynomials at 3.