1) Let $I, J$ be ideals in a ring $R$. Show that $I + J = \{ r \in R \mid r = x + y, x \in I, y \in J \}$ is an ideal. If $(n), (m)$ are two principal ideals in $\mathbb{Z}$, what is $(n) + (m)$?

Solution: The second part. Let $d$ be the greatest common divisor of $n$ and $m$. Since $(n) + (m)$ is the set of all integers $xn + ym$, and these elements are multiples of $d$, we have $(n) + (m) \subseteq (d)$. Since $xn + ym = d$ has a solution in $x$ and $y$, the equality $(n) + (m) = (d)$ holds.

2) Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be the map defined by $f(x) \mapsto f(1 + i)$. Let $I$ be the kernel of $\varphi$. Prove that $I$ is principal, i.e. find a generator $g(x)$ and prove that any element in $I$ is a multiple of $g(x)$.

Solution: $g(x) = (x - (1 + i))(x - (1 - i)) = x^2 - 2x + 2$. Let $f(x) \in \mathbb{Z}[x]$. Then, since $g(x)$ is monic,

$$f(x) = h(x)g(x) + ax + b$$

for some $h(x) \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$. If $f(x) \in I$ then, after substituting $x = 1 + i$ in the above equation, we get $0 = a(1 + i) + b$. Since $1 + i$ and $1$ are linearly independent over $\mathbb{Q}$, $a = b = 0$. Thus $f(x)$ is a multiple of $g(x)$.

3) What are the maximal ideals in $\mathbb{R}[x]/(x^2 - 3x + 2)$? In $\mathbb{R}[x]/(x^2 + x + 1)$?

Solution: Maximal ideals in a quotient ring $R/I$ correspond to maximal ideals in $R$ containing $I$. Apply this to $R = \mathbb{R}[x]$. In this ring any ideal $I$ is principal, $I = (f(x))$. Then $J = (g(x))$ contains $I$ if and only if $f(x)$ is a multiple of $g(x)$. Furthermore, $J$ is maximal if and only if $g(x)$ is an irreducible polynomial. Since $x^2 - 3x + 2 = (x - 1)(x - 2)$, the first ring has two maximal ideals generated by $x - 1$ and $x - 2$, respectively. On the other hand, $x^2 + x + 1$ is irreducible, so only the 0 ideal is maximal. (This ring is a field isomorphic to $\mathbb{C}$.)

4) Is the ring $\mathbb{Z}/10\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$? Is $\mathbb{Z}/8\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$?

Solution: The map $x \mapsto (x, x)$ defines a homomorphism from $\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. The kernel is $\mathbb{Z}/10\mathbb{Z}$, yielding an isomorphism by the first isomorphism theorem. For the second part, writing a ring as a product is the same as finding non-trivial idempotents. But $\mathbb{Z}/8\mathbb{Z}$ has no idempotents except 0 and 1, by a simple check.

5) Determine the ring $\mathbb{Z}[x]/(2x + 1, 10)$.

Solution: The idea is to divide $2x + 1$ by 10. This can be done provided we take an appropriate multiple of $2x + 1$:

$$5(2x + 1) = 10x + 5.$$
Thus $5 \in (2x + 1, 10)$ and the generator 10 can be replaced by 5. It follows that

$$\mathbb{Z}[x]/(2x + 1, 10) \cong \mathbb{Z}[x]/(2x + 1, 5) \cong \mathbb{F}_5[x]/(2x + 1) \cong \mathbb{F}_5$$

where the last isomorphism is obtained by evaluating polynomials at 2, the root of $2x + 1$ in $\mathbb{F}_5$. 