YOUR NAME:

1) (5 pts)
a) Sketch the solid \( W \) bounded by the planes \( x = 1, y = x + 1, y = -x - 1, z = 0 \) and \( z = 2 \).
**Hint:** sketch first the base of the solid in \((x, y)\)-plane.
b) Evaluate \( \int \int \int_W z \, dz \, dy \, dx \).

**Solution:** The lines \( x = 1, y = x + 1, y = -x - 1 \) bound a triangle with vertices \((-1, 0), (1, 2)\) and \((1, -2)\). The solid \( W \) is a triangular prism with height 2.

\[
\int \int \int_W z \, dz \, dy \, dx = \int_{-1}^{1} \int_{-x-1}^{x+1} \int_{0}^{2} z \, dz \, dy \, dx = 8.
\]

2) (5 pts)
a) Evaluate \( \int_{0}^{1} \int_{x^2}^{1} y \, dy \, dx \).
b) Sketch the region of integration, then change the order of integration and evaluate again.

**Solution:** a) The integral is \(2/5\). b)

\[
\int_{0}^{1} \int_{x^2}^{1} y \, dy \, dx = 2/5.
\]

3) (5 pts) Sketch the region of integration of \( \int_{0}^{1-x} \int_{0}^{1-x-y} dz \, dy \, dx \)

**Solution:** It is a pyramid with vertices \((0,0,0), (1,0,0), (0,1,0)\) and \((0,0,1)\).
4) (5 pts) Consider the paraboloid \( z = 1 - x^2 - y^2 \). Find the area of its part sitting above the \((x, y)\)-plane. Hint: use polar coordinates.

Solution: The paraboloid intersects the \((x, y)\)-plane in the unit circle \( C := x^2 + y^2 = 1 \). Thus the area is equal to
\[
A = \int \int_C \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy.
\]
Since \( z_x = -2x \) and \( z_y = -2y \), \( z_x^2 + z_y^2 = 4x^2 + 4y^2 = 4r^2 \). Thus, in polar coordinates
\[
A = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \cdot r \, dr \, d\theta.
\]

5) (5 pts) Let \( P \) be a parallelogram with vertices \((0,0), (1,1), (-1,1) \) and \((0,2)\). Compute \( \int \int_P x^2 + y^2 \, dy \, dx \) using a linear change of variable which transforms the unit square into \( P \).

Solution: The linear map is
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]
i.e. \( x = u - v \) and \( y = u + v \). Geometrically, this linear map is a counterclockwise rotation by 90 degrees, followed by a homotety by \( \sqrt{2} \). The Jacobian is 2. Since \( x^2 + y^2 = (u-v)^2 + (u+v)^2 = 2u^2 + 2v^2 \),
\[
\int \int_P x^2 + y^2 \, dy \, dx = \int_0^1 \int_0^1 (2u^2 + 2v^2) \cdot 2 \, dudv = \frac{8}{3}.
\]