

MATH 2210 - EXAM III

YOUR NAME:

1) (5 pts)

a) Sketch the solid  $W$  bounded by the planes  $x = 1$ ,  $y = x + 1$ ,  $y = -x - 1$ ,  $z = 0$  and  $z = 2$ .  
Hint: sketch first the base of the solid in  $(x, y)$ -plane.

b) Evaluate

$$\int \int \int_W z dz dy dx.$$

Solution: The lines  $x = 1$ ,  $y = x + 1$ ,  $y = -x - 1$  bound a triangle with vertices  $(-1, 0)$ ,  $(1, 2)$  and  $(1, -2)$ . The solid  $W$  is a triangular prism with height 2.

$$\int \int \int_W z dz dy dx = \int_{-1}^1 \int_{-x-1}^{x+1} \int_0^2 z dz dy dx = 8.$$

2) (5 pts)

a) Evaluate

$$\int_0^1 \int_{x^2}^1 y dy dx.$$

b) Sketch the region of integration, then change the order of integration and evaluate again.

Solution: a) The integral is  $2/5$ . b)

$$\int_0^1 \int_0^{\sqrt{y}} y dx dy = 2/5.$$

3) (5 pts) Sketch the region of integration of

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx$$

Solution: It is a pyramid with vertices  $(0,0,0)$ ,  $(1,0,0)$ ,  $(0,1,0)$  and  $(0,0,1)$ .

4) (5 pts) Consider the paraboloid  $z = 1 - x^2 - y^2$ . Find the area of its part sitting above the  $(x, y)$ -plane. Hint: use polar coordinates.

Solution: The paraboloid intersects the  $(x, y)$ -plane in the unit circle  $C := x^2 + y^2 = 1$ . Thus the area is equal to

$$A = \int \int_C \sqrt{1 + z_x^2 + z_y^2} \, dx dy.$$

Since  $z_x = -2x$  and  $z_y = -2y$ ,  $z_x^2 + z_y^2 = 4x^2 + 4y^2 = 4r^2$ . Thus, in polar coordinates

$$A = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \cdot r \, dr d\theta.$$

5) (5 pts) Let  $P$  be a parallelogram with vertices  $(0,0)$ ,  $(1,1)$ ,  $(-1,1)$  and  $(0,2)$ . Compute  $\int \int_P x^2 + y^2 \, dy dx$  using a linear change of variable which transforms the unit square into  $P$ .

Solution: The linear map is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

i.e.  $x = u - v$  and  $y = u + v$ . Geometrically, this linear map is a counterclockwise rotation by 90 degrees, followed by a homotety by  $\sqrt{2}$ . The Jacobian is 2. Since  $x^2 + y^2 = (u - v)^2 + (u + v)^2 = 2u^2 + 2v^2$ ,

$$\int \int_P x^2 + y^2 \, dy dx = \int_0^1 \int_0^1 (2u^2 + 2v^2) \cdot 2 \, dudv = \frac{8}{3}.$$