MATH 5310 SECOND EXAM - SOLUTIONS

1) (5 pts) Consider the system of linear equations

$$\begin{pmatrix} 3 & 2 & -2 \\ 3 & 3 & -3 \\ 12 & 12 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Compute the number of solutions in $\mathbb{F}_p$. You do not need to write down the solutions.

Solution: Using row/column reduction

$$\begin{pmatrix} 3 & 2 & -2 \\ 3 & 3 & -3 \\ 12 & 12 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 4 & 11 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 15 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

So the rank of the matrix is 1 if $p = 3$, 2 if $p = 5$ and 3 otherwise. Hence the kernel has the dimension 2, 1, and 0 respectively. Hence the number of solutions is 3^2, 5 and 1, respectively.

2) (5 pts) Let T be a linear transformation on an n-dimensional vector space V. Let $v \in V$ be a vector such that $T^{n-1}(v) \neq 0$ and $T^n(v) = 0$, for some positive integer n. We have shown that $v, T(v), \ldots, T^{n-1}(v)$ is a basis of V. Write down the matrix of T in this basis.

Solution: For $n = 2, 3, 4$, you get the idea...

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3) (5 pts) Which of the following subsets of $\mathbb{R}^2$ is a (real) subspace?

(1) $U = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\}$.
(2) $U = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}$.
(3) $U = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$.

Solution: (1) No. It is not closed under scalar multiplication, for example, take $(1, 1) \in U$ then $\sqrt{2}(1, 1) = (\sqrt{2}, \sqrt{2}) \notin U$. (2) No. It is not closed under scalar multiplication: $(1, 1) \in U$ but $(-1)(1, 1) = (-1, -1) \notin U$. (3) Yes, because it is the kernel of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$, $T(x, y) = x + y$.

4) (5 pts) Let F be the set of all complex numbers of the form $a + bi$ where $a, b \in \mathbb{Q}$. Prove that F is a field.

Solution: It is clear that F is a group with respect to addition and closed under multiplication. So one just needs to find a multiplicative inverse for a nonzero $a+bi$, i.e. a or b is non-zero. One can use the explicit formula for the inverse of complex numbers, or solve $(a+bi)(x+yi) = 1+0\cdot i$ which is a system of two linear equations $ax - by = 1$ and $bx + ay = 0$ in rational numbers. This system always has a solution since the determinant is $a^2 + b^2 \neq 0$ for a non-zero $a + bi$.

5) (5 pts) The group $GL_2(\mathbb{F}_p)$ of all 2×2 invertible matrices with coefficients in the finite field $\mathbb{F}_p$ has the order $(p^2-1)(p^2-p)$. Let $SL_2(\mathbb{F}_p)$ be the subgroup consisting of all matrices of determinant 1 i.e. $SL_2(\mathbb{F}_p)$ is a kernel of the homomorphism $\det : GL_2(\mathbb{F}_p) \to \mathbb{F}_p^\times$. Compute the order of $SL_2(\mathbb{F}_p)$. Hint: $GL_2(\mathbb{F}_p)$ is a disjoint union of fibers of det.

Solution: $GL_2(\mathbb{F}_p)$ is a union of fibers, and each fiber is a coset of the kernel $SL_2(\mathbb{F}_p)$. Hence the order of $GL_2(\mathbb{F}_p)$ is equal to the order of $SL_2(\mathbb{F}_p)$ times the number of fibers. The homomorphism is surjective since

$$\det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$$

hence the number of fibers is $p-1$. Thus the order of $SL_2(\mathbb{F}_p)$ is $(p-1)p(p+1)$.