1) Let $a$ and $b$ be two elements in a group $G$. Assume that $a$ has order 11 and $a^7b = ba^7$. Prove that $ab = ba$.

Solution: The reason why this works is that, since 7 is relatively prime to 11, the set of all powers of $a^7$ is the same as the set of all powers of $a$. (Check this.) Since $a^7$ commutes with $b$, any power of $a^7$ commutes with $b$. But $(a^7)^8 = a^{56} = (a^{11})^5 a = a$, thus $a$ commutes with $b$.

2) Let $G$ be an abelian group. Show that $H = \{x \in G \mid x^2 = e\}$ is a subgroup. Here $e$ is the identity element in $G$.

Solution: We need to show that $H$ is closed under multiplication and taking inverse. If $x$ and $y$ are in $H$, then $x^2 = e$ and $y^2 = e$. Multiplying this two equations gives $x^2y^2 = e \cdot e = e$. Since $G$ is abelian, $x^2y^2 = xyxy = (xy)^2$, thus $(xy)^2 = e$, and this means that $xy$ is in $H$, i.e. $H$ is closed under multiplication. The set $H$ is closed under taking inverse since $x^2 = e$ implies that $x$ is its own inverse.

Alternatively, $\varphi(x) = x^2$ is a homomorphism, by the following problem, and $H$ is the kernel of $\varphi$. Hence it is a subgroup.

3) Let $G$ be a group and $\varphi : G \to G$ the map defined by $\varphi(x) = x^2$. Show that $\varphi$ is a homomorphism if and only if $G$ is an abelian group.

Solution: Note that $\varphi(x)\varphi(y) = x^2y^2 = xxyy$, and $\varphi(xy) = (xy)^2 = xyxy$. Thus $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x$ and $y$, i.e. $\varphi$ is a homomorphism, if and only if $xyxy = xxyy$, but this is equivalent to $yx = xy$ (cancel one $x$ and one $y$) for all $x$ and $y$ i.e. $G$ is commutative.

4) Let $G$ be a group and $\sim$ be the relation such that $x \sim y$ if there exists $g$ in $G$ such that $x = gyy^{-1}$. Prove that $\sim$ is an equivalence relation.

Solution:

Reflexivity. We need to show that $x \sim x$ for all $x \in G$, i.e. for every $x$ we need to find $g$ such that $x = gxx^{-1}$. Obviously, we can pick $g = e$, the identity.

Symmetry. If $x \sim y$ i.e. $x = gyy^{-1}$ for some $g$ in $G$ then $y = g^{-1}yg$ i.e. $y \sim x$.

Transitivity. If $x \sim y$ and $y \sim z$ i.e. $x = gyy^{-1}$ for some $g$ in $G$ and $y = hzh^{-1}$ for some $h$ in $G$ then, by substituting $y = hzh^{-1}$ into the first equation and using that $h^{-1}g^{-1} = (gh)^{-1}$, we get $x = (gh)z(gh)^{-1}$ i.e. $x \sim z$. 

1
5) Compute the center of the group of matrices (with real entries) of the following form:

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}.
\]

Hint: compute first the centralizers in two special cases when either \(x\) or \(y\) is 1 and other two variables are 0.

Solution: Note first that

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

holds only if \(x = 0\). Similarly,

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

holds only if \(y = 0\). Finally, one easily checks that each matrix with \(x = 0\) and \(y = 0\) is in the center. Thus

\[
Z = \left\{ \begin{pmatrix}
1 & 0 & z \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} : z \in \mathbb{R} \right\}.
\]