FIRST EXAM, MATH 5310 - SOLUTIONS

1) Let a and b be two elements in a group G. Assume that a has order 12 and $a^5b = ba^5$. Prove that $ab = ba$.

Solution: The reason why this works is that, since 5 is relatively prime to 12, the set of all powers of a^5 is the same as the set of all powers of a. (Check this.) Since a^5 commutes with b, any power of a^5 commutes with b. But $(a^5)^5 = a^{25} = (a^{12})^2 a = a$, thus a commutes with b.

2) Let $f : \mathbb{R}^+ \to \mathbb{C}^\times$ be the map defined by $f(x) = e^{ix}$. Prove that f is a homomorphism and determine its kernel and the image.

Solution: $f(x + y) = e^{i(x+y)} = e^{ix} \cdot e^{iy} = f(x)f(y)$, by the property of the exponential function, thus f is a homomorphism. Since

$$f(x) = e^{ix} = \cos x + i \sin x$$

(Euler's formula) and $\cos^2 x + \sin^2 x = 1$, the image is the unit circle. The kernel is $2\pi\mathbb{Z}$. (The map f winds the line onto the circle with the period 2π.)

3) Let G be a group and $\varphi : G \to G$ the map defined by $\varphi(x) = x^2$. Show that φ is a homomorphism if and only if G is an abelian group.

Solution: Note that $\varphi(x)\varphi(y) = x^2y^2 = xxyy$, and $\varphi(xy) = (xy)^2 = xyxy$. Thus $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y, i.e. φ is a homomorphism, if and only if $xyxy = xxyy$, but this is equivalent to $yx = xy$ (cancel one x and one y) for all x and y i.e. G is commutative.

4) Let G be a group and $\sim$ be the relation such that $x \sim y$ if there exists g in G such that $x = gyg^{-1}$. Prove that $\sim$ is an equivalence relation. Let G be a group and H a subgroup if index 2. Show that H is normal. Hint: consider $G \setminus H$.

Solution:

Reflexivity. We need to show that $x \sim x$ for all $x \in G$, i.e. for every x we need to find g such that $x = gxg^{-1}$. Obviously, we can pick $g = e$, the identity.

Symmetry. If $x \sim y$ i.e. $x = gyg^{-1}$ for some g in G then $y = g^{-1}yg$ i.e. $y \sim x$.

Transitivity. If $x \sim y$ and $y \sim z$ i.e. $x = gyg^{-1}$ for some g in G and $y = hzh^{-1}$ for some h in G then, by substituting $y = hzh^{-1}$ into the first equation and using that $h^{-1}g^{-1} = (gh)^{-1}$, we get $x = (gh)z(gh)^{-1}$ i.e. $x \sim z$.

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5) Compute the center of the group of matrices (with real entries) of the following form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Hint: compute first the centralizers in two special cases when either x or y is 1 and other two variables are 0.

Solution: Note first that

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

holds only if $x = 0$. Similarly,

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

holds only if $y = 0$. Finally, one easily checks that each matrix with $x = 0$ and $y = 0$ is in the center. Thus

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$