ALGEBRA PROBLEMS - NOVEMBER 2007

1) Show that there are no simple groups of order 132.

2) Let G be a finite non abelian group of order p^3 where p is a prime number. If Z is the center of G, show that G/Z is isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

3) Let A be a rational 3×3 matrix such that $A^8 = I$. Show that $A^4 = I$. Hint: elementary divisors of A are rational polynomials.

4) Show that $\Phi_{3^n}(x) = x^{2 \cdot 3^{n-1}} + x^{3^{n-1}} + 1$ for every positive integer n. Use Eisenstein's criterion to show that $\Phi_{3^n}(x)$ is irreducible.

5) Let E be the splitting field of $x^p - x - 1 \in \mathbb{F}_p[x]$. Show that the degree of E over $\mathbb{F}_p$ is p. Hint: what is the order of the Frobenius automorphism?

6) Let $E = F(\alpha)$ be an extension of the field F of degree n.
   (1) If n is prime to 2, show that $E = F(\alpha^2)$.
   (2) Give an example in which n is prime to 3, but $F(\alpha^3) \neq E$.

7) Prove that there exists a Galois extension K of $\mathbb{Q}$ with the Galois group isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}.$$

You may use that the Galois group of $\mathbb{Q}(\zeta_n)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.

8) Let M be a non-trivial, finitely generated $\mathbb{Z}$-submodule in $\mathbb{Q}$. Show that $M = \frac{1}{n}\mathbb{Z}$ for some integer n. In particular, M is isomorphic to $\mathbb{Z}$.

9) Show that the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n=2}^{\infty}(\mathbb{Z}/n\mathbb{Z})$ is non trivial. Hint: if $1 \otimes (1, 1, \ldots) = 0$ then this holds in a tensor product $M \otimes_{\mathbb{Z}} \prod_{n=2}^{\infty}(\mathbb{Z}/n\mathbb{Z})$ for some finitely generated module M in $\mathbb{Q}$.