We recall some basic definitions and properties.

**Fundamental Theorem of Algebra.** Every nonzero polynomial of degree \( n \) with complex coefficients has precisely \( n \) complex roots \( a_1, a_2, \ldots, a_n \) and can be factored as

\[
p(x) = c(x - a_1)(x - a_2) \cdots (x - a_n).
\]

For example, the complex roots of \( x^n - 1 = 0 \) (\( n \)-th roots of 1) form a regular \( n \)-gon in complex plane, and can be written as

\[
e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 1, \ldots, n.
\]

**Polynomial Interpolation.** For any sequence \( a_1, a_2, \ldots, a_n \) of \( n \) complex numbers and \( n \) distinct points \( x_1, x_2, \ldots, x_n \) in \( \mathbb{C} \) there exists a unique polynomial of degree at most \( n - 1 \) such that

\[
p(x_j) = a_j, \quad j = 0, 1, \ldots, n.
\]

This polynomial is easily constructed using Lagrange's interpolation:

\[
P(x) = \sum_{j=0}^{n} a_j \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)}.
\]

**Division Algorithm.** If \( f \) and \( g \neq 0 \) are two polynomials with coefficients in a field \( F \), then there exist unique polynomials \( q \) and \( r \) such that

\[
f = qg + r.
\]

here either \( r = 0 \) or \( \deg(r) < \deg(g) \). The situation with polynomials with coefficients in a ring is more delicate. However, if \( f \) and \( g \) have integer coefficients, and \( g \) is monic (the first non-zero coefficient is 1), then \( q \) and \( r \) also have integer coefficients.

One can use the division algorithm to show that \( F[x] \) is a unique factorization domain. That is, every polynomial \( p(x) \) can be factored into irreducible polynomials \( p(x) = q_1(x) \cdot \ldots \cdot q_m(x) \) where \( q_i(x) \) are uniquely
determined up to a permutation and multiplication of each $q_i(x)$ by a non-zero scalar $c_i$ such that $c_1 \cdot \ldots \cdot c_m = 1$. Once we know that $F[x]$ is a unique factorization domain, it can be shown that $F[x_1, \ldots, x_n]$, the ring of polynomials in $n$ variables is also a unique factorization domain.

**Invariant Polynomials.** Here we discuss polynomials in $n$ variables $x_1, \ldots, x_n$.

A polynomial $p$ is said to be invariant if it does not change under any permutation of $n$ variables. Every invariant polynomial in $x_1, \ldots, x_n$ is a polynomial in elementary symmetric polynomials $c_1, c_2, \ldots, c_n$ which are defined by

$$(x - x_1)(x - x_2)\cdots(x - x_n) = x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n.$$ 

For example, if $n = 3$ then there are three elementary symmetric functions and they are

$$\begin{cases} 
    c_1 = -(x_1 + x_2 + x_3) \\
    c_2 = x_1x_2 + x_1x_3 + x_2x_3 \\
    c_3 = -x_1x_2x_3.
\end{cases}$$

The polynomial $S_2 = x_1^2 + x_2^2 + x_3^2$ is symmetric and in terms of $c_i$’s it can be written as

$$S_2 = c_1^2 - 2c_2.$$ 

This formula is a special case of **Newton’s formulas for power sums**. More precisely, let $m$ be a positive integer. The $m$-th power sum in $x_1, \ldots, x_n$ is

$$S_m = x_1^m + x_2^m + \cdots + x_n^m.$$ 

Then

$$\begin{align*}
S_1 + c_1 &= 0 \\
S_2 + c_1S_1 + 2c_2 &= 0 \\
S_3 + c_1S_2 + c_2S_1 + 3c_3 &= 0 \\
&\quad \vdots \\
S_n + c_1S_{n-1} + \cdots + c_{n-1}S_1 + nc_n &= 0 \\
S_m + c_1S_{m-1} + \cdots + c_{m-n}S_{m-n} &= 0
\end{align*}$$

where the last equality holds for $m > n$. Notice that the formulas imply not only that $S_k$ can be expressed in terms of $c_k$ but, conversely, that $c_k$ can be expressed in terms of $S_k$ for $k \leq n$. In particular, it follows that any symmetric polynomial in variables $x_1, \ldots, x_n$ can be expressed as a polynomial in $S_k$ for $k \leq n$, as well.
Palindromic polynomials. The equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ is called palindromic if $a_j = a_{n-j}$ for all $j$. If $n$ is even, then by substitution $z := x + 1/x$ reduces to an equation of degree $n/2$. After finding solutions $z_j, j = 1, \ldots z_{n/2}$, the solutions of the original equation are found by solving $x + \frac{1}{x} = z_j$ for all $j$.

The following is an important observation. If $a$ is a real number then solutions of

$$x + \frac{1}{x} = a$$

are real if $|a| \geq 2$ and complex numbers on the unit circle ($|x| = 1$) if $|a| \leq 2$.

Cubic equation. The equation $x^3 + px + q = 0$ has three solutions which are described as follows: Put

$$R = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

and

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$$

Then the three solutions are

$$\{ A + B, A\rho + B\rho^2, A\rho^2 + B\rho \}$$

where $\rho = e^{2\pi i/3}$ is a cubic root of 1. The number $R$ is essentially the discriminant of the cubic. More precisely,

$$108R = -(x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2$$

where $x_1, x_2$ and $x_3$ are three roots of the cubic polynomial. Assume that $p$ and $q$ are real. If all three roots are real then $R < 0$. If one root is real and the other two are complex conjugates of each other, then $R > 0$. Thus calculating the discriminant is a quick way to see whether a cubic has one or three real roots.
Exercises:

1. Find the reminder when \( x^{81} + x^{49} + x^{25} + x^9 + x \) is divided by a) \( x^3 - x \),
   b) \( x^2 + 1 \), c) \( x^2 + x + 1 \).

2. Let \( p \) be a non-constant polynomial with integral coefficients. If \( p(k) = 0 \) for four distinct integers \( k \), prove that \( p(k) \) is composite for every integer \( k \).

3. Factor \((a + b + c)^3 - (a^3 + b^3 + c^3)\).

4. Find \( a \) if \( a \) and \( b \) are integers such that \( x^2 - x - 1 \) is a factor of \( ax^{17} + bx^{16} + 1 \).

5. Find the unique polynomial of degree \( n \) such that
   \[ p(j) = 2^j \]
   for \( j = 0,1,\ldots,n \).

6. Find the unique polynomial \( p(x) \) of degree \( n \) such that
   \[ p(j) = \frac{1}{1+j} \]
   for \( j = 0,1,\ldots,n \).

7. A polynomial \( p_n \) of degree \( n \) satisfies \( p_n(k) = F_k \) for \( k = n + 2, n + 3, \ldots, 2n + 2 \), where \( F_k \) are Fibonacci numbers. Show that \( p_n(2n+3) = F_{2n+3} - 1 \).

8. If
   \[
   \begin{align*}
   x + y + z &= 1 \\
   x^2 + y^2 + z^2 &= 2 \\
   x^3 + y^3 + z^3 &= 3
   \end{align*}
   \]
   find \( x^4 + y^4 + z^4 \).

9. Complex solutions of \( x^n - 1 = 0 \) are \( \zeta^k, k = 1,\ldots,n \) where \( \zeta = e^{2\pi i/n} \).
   Show that
   \[
   \sum_{k=1}^{n} \zeta^{dk} = \begin{cases} 
   n & \text{if } d \text{ is a multiple of } n \\
   0 & \text{otherwise}
   \end{cases}
   \]
10. Find a cubic equation whose roots are cubes of the roots of \(x^3 + ax^2 + bx + c = 0\).

11. Let \(x_1, x_2, \ldots, x_n\) be the roots of a polynomial \(P(x) = x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \cdots\) of degree \(n\). The number

\[
\prod_{i \neq j} (x_i - x_j)
\]

is called the discriminant of \(P\). It is symmetric in variables \(x_i\), so it can be expressed as a polynomial in the coefficients of \(P\). Do this for \(n = 2\) and \(n = 3\). If \(n = 3\), assume that the polynomial is \(x^3 + px + q\).

12. Find all values of the parameter \(a\) such that all roots of the equation

\[x^6 + 3x^5 + (6 - a)x^4 + (7 - 2a)x^3 + (6 - a)x^2 + 3x + 1 = 0\]

are real.

13. The roots of the fifth degree equation

\[x^5 - 5x^4 - 35x^3 + \ldots\]

form an arithmetic sequence. Find the roots.

14. Let \(G_n = x^n \sin nA + y^n \sin nB + z^n \sin nC\) where \(x, y, z, A, B, C\) are real numbers such that \(A + B + C\) is a multiple of \(\pi\). Show that if \(G_1 = G_2 = 0\) then \(G_n = 0\) for all positive \(n\).

15. If

\[
\begin{align*}
x + y + z &= 3 \\
x^2 + y^2 + z^2 &= 25 \\
x^4 + y^4 + z^4 &= 209
\end{align*}
\]

find \(x^{100} + y^{100} + z^{100}\).

16. (B1 2005) Find a non-zero polynomial \(P(x, y)\) such that \(P([a], [2a]) = 0\) for all real numbers for all real numbers \(a\). (Note: \([a]\) is the greatest integer less then or equal to \(a\).)

17. (A3 2001) Find values for the integer \(m\) such that

\[P_m = x^4 - (2m + 4)x^2 + (m - 2)^2\]

is a product of two non-constant polynomials with integer coefficients?
Hints and solutions

1. (part c)). The roots of $x^2 + x + 1$ are $\rho$ and $\bar{\rho} = \rho^2$, two cubic roots of 1:

$$\rho = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

Thus

$$x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^2 + x + 1) = ax + b$$

gives

$$1 + \rho + \rho + 1 = a\rho + b$$

which implies that $a = 2$ and $b = 2$.

2. Hint: Let $a, b, c, d$ be the four integral roots. Then

$$p(x) = (x-a)(x-b)(x-c)(x-d)q(x)$$

where $q(x)$ has integral coefficients. If $x$ is an integer, then $(x-a)(x-b)(x-c)(x-d)$ is composite.

3. Hint: Use formulas for a difference and a sum of two cubes. Then

$$(a + b + c)^3 - a^3 = (b + c)(\cdots)$$

and

$$b^3 + c^3 = (b + c)(\cdots).$$

It follows that $(b + c)$ divides our polynomial. By symmetry, $a + c$ and $a + b$ also divide our polynomial, as well. Thus

$$(a + b + c)^3 - (a^3 + b^3 + c^3) = k(a + b)(a + c)(b + c)$$

for a constant $k$. Now put $a = b = c = 1$ to find that $k = 1$.

4. Solution: The solutions of $x^2 - x - 1 = 0$ are

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \bar{\alpha} = \frac{1 - \sqrt{5}}{2}.$$

Thus $ax^{17} + bx^{16} + 1 = q(x)(x^2 - x - 1)$ gives

$$a\alpha^{17} + b\alpha^{16} + 1 = 0$$

$$a\bar{\alpha}^{17} + b\bar{\alpha}^{16} + 1 = 0$$
which is a $2 \times 2$ system with $a$ and $b$ as unknowns. Since $\alpha \bar{\alpha} = -1$, the determinant of this system is equal to $\alpha - \bar{\alpha} = \sqrt{5}$. Thus, using the Cramer’s rule, the solution is

$$a = \frac{(1 + \sqrt{5})^{16} - (1 - \sqrt{5})^{16}}{2^{16}\sqrt{5}}$$

which we recognize as the 16-th Fibonacci number.

5. Hint: Use the Lagrange’s interpolation.

6. Hint: Consider $q(x) = (1 + x)p(x) - 1$. Then

$$q(x) = cx(x - 1)\ldots(x - j)$$

Now pick $c$ so that $q(x) + 1$ is divisible by $1 + x$.

7. Hint: Notice that for $k = n + 1, n + 2, \ldots, 2n$

$$p_n(k + 2) - p_n(k + 1) = F_{k+2} - F_{k+1} = F_k$$

so $p_{n-1}(x) = p_n(x + 2) - p_n(x + 1)$. Use this to do induction.

1) Base of induction: Let $n = 1$. Since $p_1(3) = 2$ and $p_1(4) = 3$ then $p_1(x) = x + 1$. Thus

$$p_1(5) = 4 = F_5 - 1.$$  

2) Step of induction: Assume that $p_{n-1}$ satisfies $p_{n-1}(2n+1) = F_{2n+1} - 1$. Thus

$$p_n(2n+3) = p_n(2n+2) + p_{n-1}(2n+1) = F_{2n+2} + F_{2n+1} - 1 = F_{2n+3} - 1.$$  

8. Elementary exercise in symmetric polynomials.

9. You are asked to compute the power sums for the roots of the polynomial $x^n - 1 = 0$. Note that all $c_i = 0$ except $c_n = 1$.

10. Hint: the coefficients of the desired cubic are also symmetric functions in roots of the original cubic.

11. If $n = 2$, write the polynomial in a more familiar $x^2 + px + q = 0$.

$$(x_1 - x_2)(x_2 - x_1) = 2x_1x_2 - (x_1^2 + x_2^2) = 4x_1x_2 - (x_1 + x_2)^2 = 4q - p^2.$$  

Note that this expression appears in the quadratic formula! If $n = 3$, it takes a bit more to do the problem.
12. Hint: The polynomial is palindromic so it can be reduced to the degree 3. Next, the solutions of \( x + \frac{1}{x} = b \) are real iff \( |b| \geq 2 \).

13. Hint: The 5 roots are \( a, a+d, a+2d, a+3d \) and \( a+4d \) for two numbers \( a \) and \( b \). The first two symmetric polynomials are known. This gives
\[
5a + 10d = 5 \quad \text{and} \quad 10a^2 + 40ad + 35d^2 = -35.
\]

14. Solution: Put \( X = xe^{iA}, Y = ye^{iB} \) and \( Z = ze^{iC} \). Then \( G_n \) is the imaginary part of the \( n \)-th power sum
\[
S_n = X^n + Y^n + Z^n.
\]
Since \( G_1 = G_2 = 0 \) the first two elementary symmetric polynomials in \( X, Y, Z \) are real. Since \( XYZ = xyz e^{i(A+B+C)} = \pm xyz \),
\[(\text{here we are using that } A+B+C \text{ is a multiple of } \pi) \text{ the third elementary symmetric polynomial is also real. Since all } S_n \text{ can be expressed as polynomials with integer coefficients of the three elementary symmetric polynomials, they have to be real.}\]

15. Hint: (I am not so sure, though.) Note that any symmetric polynomial is a polynomial in \( S_1, S_2 \) and \( S_3 \). By degree considerations, \( S_4 \) must be linear in \( S_3 \), so we can certainly figure out \( S_3 \), and therefore \( x, y \) and \( z \) as solutions of a cubic?

16. Hint: This problem is very easy. What is the possible difference \( [2a] - 2[a] \)?

17. Hint: Find the roots.