ON LOCAL LIFTS FROM $G_2(\mathbb{R})$ TO $\text{Sp}_6(\mathbb{R})$ AND $F_4(\mathbb{R})$

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ABSTRACT

Let $G_2(\mathbb{R}) \times \text{Sp}_6(\mathbb{R})$ and $G_2(\mathbb{R}) \times F_4(\mathbb{R})$ be split dual pairs in split $E_7(\mathbb{R})$ and $E_8(\mathbb{R})$, respectively. It is known that the exceptional correspondences for these dual pairs are functorial on the level of infinitesimal characters. In this paper we show that these dual pair correspondences are functorial for the minimal $K$-types of principal series representations.

1. Split real groups of type $E_n$

The Cartan decomposition for split real groups of type $E_n$ can be described by Jordan algebras of rank 4, as it has been shown by Kostant and Brylinski in [3]. To this end, let $J = J_n(Q)$ be a Jordan algebra of $n \times n$-hermitian symmetric matrices over a composition algebra $Q$. To each Jordan algebra, $J$, one can attach a simple Lie algebra $\mathfrak{k} = \mathfrak{k}(J)$ with a short $\mathbb{Z}$-filtration

$$\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

such that $\mathfrak{k}_1 \cong J$. The algebra $\mathfrak{k}$ has $n$ strongly orthogonal roots $\alpha_1, \ldots, \alpha_n$ corresponding to the diagonal entries of $J$. Let

$$\psi = \frac{1}{2}(\alpha_1 + \cdots + \alpha_n)$$

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Of special interest to us is the case \( n = 4 \), in which case \( \langle \psi, \psi \rangle = 2 \). Let \( p \) be the irreducible \( k \)-module of highest weight \( \psi \). Then the exceptional Lie algebras of type \( E_n \) have Cartan decomposition

\[
g = \mathfrak{k} \oplus p
\]

where \( p \cong V_\psi \), as a \( \mathfrak{k} \)-module, and \( \mathfrak{k} = \mathfrak{k}(J_4(Q)) \) where \( Q \) is a composition algebra over \( \mathbb{C} \) of dimension 1, 2 and 4 for \( E_6, E_7 \) and \( E_8 \), respectively. The minimal representation (the corresponding \( (g, K) \)-module) has \( K \)-types

\[
V = \oplus_{i=0}^{\infty} V_{\psi_i}
\]

This \( (g, K) \)-module corresponds to a representation of the simply connected Chevalley group of type \( E_n \). This representation is faithful except for \( E_7 \) when the center \( \mu_2 \) acts trivially.

**The Dual Pairs.** Simply connected Chevalley group \( E_n(\mathbb{R}) \) contains a split dual pair \( H \times G_2 \) where \( H \) is \( SL_3(\mathbb{R}) \), \( Sp_6(\mathbb{R}) \) and \( F_4(\mathbb{R}) \) respectively. Let \( K_1 \) and \( K_2 = SU_{2, l} \times_{\mu_2} SU_{2, s} \) denote the maximal compact subgroup of \( H \) and \( G_{2,2} \) respectively. The two factors of \( K_2 \) correspond to a pair of perpendicular roots, one long and one short, as the subscripts indicate. The possible \( K \) and \( K_1 \) are tabulated below.

<table>
<thead>
<tr>
<th>Split group</th>
<th>( K )</th>
<th>( \mathfrak{p} = V(\psi) )</th>
<th>( H )</th>
<th>( K_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_6 )</td>
<td>( Sp_6/\mu_2 )</td>
<td>( V(\varpi_4) = \wedge^4 C^8 - \wedge^2 C^8 )</td>
<td>( SL_3(\mathbb{R}) )</td>
<td>( SO_3 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( SU_8/\mu_2 )</td>
<td>( V(\varpi_4) = \wedge^4 C^8 )</td>
<td>( Sp_6(\mathbb{R}) )</td>
<td>( U_3 )</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>( Spin_{16}/\mu_2 )</td>
<td>( V(\varpi_8) )</td>
<td>( F_4 )</td>
<td>( SU_2 \times_{\mu_2} Sp_6 )</td>
</tr>
</tbody>
</table>

For root systems and weights, we follow the enumeration of Bourbaki [1].

We shall now describe how \( K_1 \times K_2 \) embeds into \( K \). Let \( \mathfrak{t}_2 = \mathfrak{sl}(2)_s + \mathfrak{sl}(2)_l \) be the complexified Lie algebra of \( K_2 \). Recall that \( \mathfrak{t} = \mathfrak{t}(J_4(Q)) \) has four strongly orthogonal roots. The long root \( \mathfrak{sl}(2)_l \) embeds as \( \mathfrak{sl}(2) \) corresponding to the root \( \alpha_1 \) and the short root \( \mathfrak{sl}(2)_s \) embeds diagonally into three \( \mathfrak{sl}(2) \) corresponding to the remaining three roots. The centralizer of \( \mathfrak{t}_2 \) in \( \mathfrak{k} \) is \( \mathfrak{t}_1 \).

**The Langlands Quotients.** Let \( \pi_1 \) and \( \pi_2 \) be irreducible Harish–Chandra modules of \( \mathfrak{h} \) and \( \mathfrak{g}_2 \) respectively. Let

\[
V_{\min} \to \pi_1 \boxtimes \pi_2
\]

be a nonzero morphism of \( (\mathfrak{h} \times \mathfrak{g}_2) \)-modules. As established in [4] (see also [7]), the infinitesimal character of \( \pi_1 \) determines the infinitesimal character of \( \pi_2 \).
and conversely. If \( xw_1 + yw_2 \) is the infinitesimal character of \( \pi_2 \), then \( \pi_1 \) has infinitesimal character \( xw_1 + (x + 3y)w_2 \), \( (x + 2y, x + y, y) \), \( xw_4 + yw_3 + \rho(\mathfrak{sl}_3) \) for \( \mathfrak{h} = \mathfrak{sl}_3, \mathfrak{sp}_6, \mathfrak{f}_4 \) respectively. We would like to refine this information in the case when \( \pi_1 \) and \( \pi_2 \) are Langlands quotients of principal series representations. More precisely, let \( B_1 = M_1A_1N_1 \) and \( B_2 = M_2A_2N_2 \) denote Borel subgroups of \( H \) and \( G_2 \), respectively. Note that \( M_1 = \mu_2^{r(H)} \), where \( r(H) \) is the rank of \( H \). Likewise, \( M_2 = \mu_2^{2} \). Let \( \sigma_i \) be a representation of \( M_i \) and let \( \lambda_i \) be a dominant weight of the Lie algebra \( \mathfrak{a}_i \). We set \( I_1(\sigma_1, \lambda_1) \) to be the Harish–Chandra module of the normalized induced representation

\[
\text{Ind}_{M_1A_1N_1}^H(\sigma_1 \otimes \mathfrak{a}^{\lambda_1}).
\]

Similarly we define \( I_2(\sigma_2, \lambda_2) \) which is a Harish–Chandra module of \( G_2 \). Next, we specify a minimal \( K_i \) type, denoted by \( \tau(\sigma_i) \), contained in the principal series \( I_1(\sigma_i, \lambda_i) \). This \( K_i \)-type depends only on the Weyl group conjugation class of \( \sigma_i \), and the restriction of \( \tau(\sigma_i) \) to \( M_i \) is a direct sum, with multiplicity one, of all characters of \( M_i \) Weyl group conjugated to \( \sigma_i \). In particular, \( \tau(\sigma_i') \) is contained in \( I_1(\sigma_i', \lambda_i) \) if and only if the characters \( \sigma_i' \) and \( \sigma_i'' \) are conjugated by the Weyl group. This minimal \( K_i \)-type is also known as the fine \( K_i \)-type and small \( K_i \)-type in [10]. The minimal \( K_i \)-type \( \tau(\sigma_i) \) is contained in the unique irreducible quotient \( J_i(\sigma_i, \lambda_i) \) of \( I_i(\sigma_i, \lambda_i) \). If \( \sigma_i \) is the trivial character of \( M_i \), then \( \tau(\sigma_i) \) is the trivial \( K_i \)-type and \( J_i(\sigma_i, \lambda_i) \) is a spherical representation. Other cases are tabulated below. These were also computed in Table 5.8 in [9].

<table>
<thead>
<tr>
<th></th>
<th>( G_2 )</th>
<th>( \text{SL}_3(\mathbb{R}) )</th>
<th>( \text{Sp}_6(\mathbb{R}) )</th>
<th>( F_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mathbb{C} \otimes \mathbb{C} )</td>
<td>( \mathbb{C} )</td>
<td>( \mathbb{C} )</td>
<td>( \mathbb{C} \otimes \mathbb{C} )</td>
</tr>
<tr>
<td></td>
<td>( \mathbb{C} \otimes S^2(\mathbb{C}^2) )</td>
<td>( \mathbb{C}^3 )</td>
<td>( \land^2\mathbb{C}^3 )</td>
<td>( \mathbb{C}^2 \otimes \mathbb{C}^0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( S^2(\mathbb{C}^2) \otimes \mathbb{C} )</td>
<td></td>
</tr>
</tbody>
</table>

Remark: Note that the list does not include two minimal \( U_3 \)-types: \( \mathbb{C}^3 \) and \( \land^3\mathbb{C}^3 \). This is because \( \mu_2 \) the center of \( E_7 \), which is also the center of \( \text{Sp}_6(\mathbb{R}) \), acts trivially on the minimal representation.

**Theorem 1:** Suppose \( V_{\text{min}} \rightarrow J_1(\sigma_1, \lambda_1) \otimes J_2(\sigma_2, \lambda_2) \) is a nonzero morphism of \((\mathfrak{h} \times \mathfrak{g}_2)\)-modules. Let \( \tau_i \) be the minimal \( K_i \)-type of \( J_i(\sigma_i, \lambda_i) \), \( i = 1, 2 \). Then \( \tau_1 \) and \( \tau_2 \) are on the same row of the above table. Moreover, if \( H = F_4 \), then representations with the minimal \( K_1 \)-type \( S^2(\mathbb{C}^2) \otimes \mathbb{C} \) do not appear as quotients of \( V_{\text{min}} \).

We remark that an irreducible spherical representation is uniquely determined by its infinitesimal character. In §3, we will discuss the situations when the
infinitesimal characters are generic. In that case we have more precise results (Theorem 6 and Theorem 9).

Since \( J_1(\sigma_1, \lambda_1) \) and \( J_2(\sigma_2, \lambda_2) \) are generated by their minimal \( K_i \)-types and \( V_{\text{min}}|_K = \sum V_K(n\lambda_0) \), the above theorem follows immediately from the following technical lemma.

**Lemma 2:** Let \( \tau_1 \) and \( \tau_2 \) be nontrivial minimal \( K_1 \)-type and \( K_2 \)-type respectively as in the third row of the above table. Then

(i) \( \text{Hom}_{K_1 \times K_2}(1_{K_1} \boxtimes \tau_2, V_n\psi) = 0 \) for all \( n \in \mathbb{N} \) and

(ii) \( \text{Hom}_{K_1 \times K_2}(\tau_1 \boxtimes 1_{K_2}, V_n\psi) = 0 \) for all \( n \in \mathbb{N} \).

Furthermore, in the case \( H = F_4 \), \( \text{Hom}_{K_1}(S^r(\mathbb{C}^2) \boxtimes S^s(\mathbb{C}^6), V_n\psi) = 0 \) unless \( r \leq s \) and \( r \equiv s \mod 2 \).

This was proven in [2] in the case \( E_6 \), and in this paper we will only deal with \( E_7 \) and \( E_8 \). Note that the group \( K_1 \times K_2 \) is much smaller than \( K \). In particular, the lemma does not follow from any of the known, classical, branching rules. We also do not develop any new branching rules. In order to illustrate the main idea consider the first case of the above Lemma. As a first step, we calculate \( K_1 \times SU_{2,l} \) invariants in \( V_n\psi \). This is done on a case by case basis and is the most difficult part of this paper. Part (i) of the lemma states that the representation \( S^2(\mathbb{C}^2) \) does not appear in the space of \( K_1 \times SU_{2,l} \) invariants. Since the highest weight of \( S^2(\mathbb{C}^2) \) is 2 it suffices to show that the dimension of the weight 4 space is the same as the dimension of the weight 2 space. In fact, using a nice trick, one can show that the dimension of the weight 2 space is the same as the dimension of the weight \(-4\) space. A similar trick works for the case (ii). In addition, in the case of \( E_8 \), we show that the \( K_1 \)-types \( S^r(\mathbb{C}^2) \boxtimes S^s(\mathbb{C}^6) \) does not appear in the minimal representation unless \( s \geq r \) and \( r \equiv s \mod 2 \). This gives severe restrictions on possible \( F_4 \)-quotients of the minimal representation.

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2. **Generic principal series representations**

In this section, we will show that if the infinitesimal character \( \lambda_1 \) of \( \pi_1 \) in (1) is generic (which we will define below), then \( \pi_i \) is isomorphic to \( I_i(\sigma_1, \lambda_1) \) (for \( i=1, 2 \)) which is irreducible.

First we recall a theorem of Speh and Vogan [8]. There is also a converse statement but we do not need it here.
Theorem 3: Let $P = MAN$ be a cuspidal parabolic subgroup of a real reductive group $G$. Let $T_M$ be a compact Cartan subgroup of $M$. Let $H_\lambda$ be a discrete series representation of $M$ with infinitesimal character $\lambda$. We consider the normalized induced principal series representation

$$\text{Ind}_{G_P}^G (H_\lambda \boxtimes a^\nu \boxtimes 1)$$

of $G$ with infinitesimal character $\gamma = (\lambda, \nu)$. Suppose $\gamma$ is nonsingular. If the principal series representation above is reducible, then there exists a non-compact root $\alpha$ of $T_M A$ such that $n = 2(\alpha, \gamma)/\alpha, \alpha)$ is a positive integer.

The above theorem leads us to the following definition: A weight $\gamma$ is called algebraically integral with respect to a root $\alpha$ if $2(\alpha, \gamma)/\alpha, \alpha)$ is an integer. In this paper, we say that the weight $\gamma$ is generic if it is not algebraically integral to any root $\alpha$ in the root system.

Lemmas 4 and 5 below follow from Theorem 3.

Lemma 4 (generic representations of $G_2(\mathbb{R})$):

- The following two statements are equivalent:
  1. The weight $\lambda_2 := x\varpi_1 + y\varpi_2$ is generic with respect to the root system of $G_2$.
  2. None of the following six numbers are integers: $x, y, x+y, x+2y, x+3y, 2x+3y$.

- If $\pi_2$ is an irreducible Harish–Chandra module of $G_2$ with infinitesimal character $\lambda_2$ satisfying either (i) or (ii) above, then $\pi_2$ is the irreducible principal series representation $I_2(\sigma_2, \lambda_2)$ for some character $\sigma_2$ of $M_2$.

Proof: Items (i) and (ii) are equivalent by considering $2(\alpha, \lambda_2)/\alpha, \alpha)$ for all roots $\alpha$ of $G_2$. Given $\pi_2$ in the lemma, then it is the Langlands quotient of $I_2(\sigma_2, \lambda_2)$ for some character $\sigma_2$ of $M_2$. By Theorem 3, $I_2(\sigma_2, \lambda_2)$ is irreducible. This proves the lemma.

Lemma 5 (generic representations of $Sp_6(\mathbb{R})$):

- The following two statements are equivalent:
  1. The weight $\lambda_1 := (x+2y, x+y, y)$ is generic with respect to the root system of $Sp_6(\mathbb{R})$.
  2. None of the following six numbers are integers: $x, y, x+y, x+2y, x+3y, 2x+3y$. 

Proof: Items (i) and (ii) are equivalent by considering $2(\alpha, \lambda_1)/\alpha, \alpha)$ for all roots $\alpha$ of $G_2$. Given $\pi_2$ in the lemma, then it is the Langlands quotient of $I_2(\sigma_2, \lambda_2)$ for some character $\sigma_2$ of $M_2$. By Theorem 3, $I_2(\sigma_2, \lambda_2)$ is irreducible. This proves the lemma.
• If $\pi_1$ is an irreducible Harish–Chandra module of $\text{Sp}_6(\mathbb{R})$ with infinitesimal character $\lambda_1$ satisfying either (i) or (ii) above, then $\pi_1$ is the irreducible principal series representation $I_1(\sigma_1, \lambda_1)$ for some character $\sigma_1$ of $M_1$.

The proof of the lemma is similar to the previous one. We will call $\pi_1$ and $\pi_2$ in the above two lemmas generic principal series representation.

In the notation of the above two lemmas, the correspondence of infinitesimal characters for the dual pair $G_2 \times \text{Sp}_6(\mathbb{R})$ is given by $\lambda_1 \leftrightarrow \lambda_2$. We can incorporate these into Theorem 1 and we have the following result.

**Theorem 6:** Suppose $\pi_1 \boxtimes \pi_2$ is a quotient of the minimal representation of $E_7$. Then $\pi_1$ is a generic spherical principal series representation of $\text{Sp}_6(\mathbb{R})$ with infinitesimal character $\lambda_1$ if and only if $\pi_2$ is a generic spherical principal series representation of $G_2$ with infinitesimal character $\lambda_2$. $lacksquare$

**Representations of split $F_4$.** Suppose $\pi_1$ is a representation of the split $F_4$ whose infinitesimal character is $\lambda_1 := x\varpi_1 + y\varpi_2 + \rho(\mathfrak{sl}_3)$ where $x$ and $y$ satisfies Lemma 4 (ii). We would like to know all possible Langlands parameters it can have. We assume that $\pi_1$ is the quotient of the principal series representation

$$ (2) \quad \text{Ind}_{MAN}^{F_4}(H_\lambda \boxtimes a' \boxtimes 1) $$

where $MAN$ is a cuspidal representation of $F_4$ and $H_\lambda$ is a discrete series representation of $M$ with Harish–Chandra parameter $\lambda$.

**Lemma 7:** Suppose $\pi_1$ satisfies the above assumptions, then the parabolic subgroup $P = MAN$ in (2) is either

(i) the Borel subgroup, or

(ii) it is the parabolic subgroup corresponding to the long simple root $\alpha_1$ or $\alpha_2$, and the connected component of $M$ is $M^0 = \text{SL}_2(\mathbb{R})$. The discrete series $H_\lambda$ of $M$ has Harish–Chandra parameter $\lambda = 1$ or 2.

**Proof:** We assume that $P$ is not the Borel subgroup. Let $\alpha$ be a simple root in $M$. Then $\lambda + \rho(M)$ is algebraically integral with respect to $\alpha$. Let $\lambda_1 := x\varpi_1 + y\varpi_2 + \rho(\mathfrak{sl}_3)$. From the consideration of infinitesimal characters, $w(\lambda_1) = (\lambda, \nu)$ for some $w$ in the Weyl group. Since $(0, \nu)$ is perpendicular to $\alpha$ and $\rho(M)$ is algebraically integral with respect to $\alpha$, we conclude that

$$ (3) \quad 2(\lambda_1, \alpha')/(\alpha', \alpha') \in \mathbb{Z} $$
where \( \alpha' = w^{-1} \alpha \). Suppose \( \alpha' \) is short root, then a check by hand shows that under the assumptions in Lemma 4(ii), (3) is impossible. This implies that \( \alpha \) is a long simple root, that is \( \alpha = \alpha_1 \) or \( \alpha_2 \).

A similar check shows that (3) holds if and only if \( \alpha' = w^{-1} \alpha \) is either \( \alpha_1, \alpha_2 \) or \( \alpha_1 + \alpha_2 \). Note that \( M^0 \) cannot be \( SL_3(\mathbb{R}) \) because it does not have discrete series representation. Hence \( M^0 = SL_2(\mathbb{R}) \) and this proves (i). The restriction of \( \lambda_1 \) to the diagonal \( H_\alpha \in \mathfrak{sl}_2 \) corresponding to \( \alpha \) is either 0, \( \pm 1 \) or \( \pm 2 \). We can remove the negative signs since \( SL_2(\mathbb{R}) \) is not a subgroup of \( M \). This proves (ii).

Next we list all the possible minimal \( K_1 \)-types of (2) satisfying the last lemma. If the parabolic subgroup is the Borel subgroup, then the minimal \( K_1 \)-types are given in the last column of Table 1. If the parabolic is not the Borel subgroup, then the \( K_1 \)-types are given in the following table.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Minimal ( K_1 )-types</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( S^4(\mathbb{C}^2) \otimes 1_{Sp_6} ) and ( S^5(\mathbb{C}^2) \otimes \mathbb{C}^6 )</td>
</tr>
<tr>
<td>2</td>
<td>( S^6(\mathbb{C}^2) \otimes 1_{Sp_6} ) and ( S^5(\mathbb{C}^2) \otimes \mathbb{C}^6 )</td>
</tr>
</tbody>
</table>

The proof is a little long but not hard so we will leave it to the reader.

**A degenerate principal series representation of \( F_4 \).** Let \( P_{12} = M_{12} \cdot (\mathbb{R}^+)^2 \cdot N_{12} \) be the (non-cuspidal) standard parabolic subgroup whose Levi factor \( M_{12} \) has simple long roots \( \{ \alpha_1, \alpha_2 \} \). Let \( \phi_i : SL_2(\mathbb{R}) \to F_4 \) be the homomorphism induced by the simple real root \( \alpha_i \). We have

\[
M_{12} = SL_3(\mathbb{R}) \times L_4
\]

where \( L_4 \) is the Klein four group generated by \( \phi_3(-1) \) and \( \phi_4(-1) \).

Let \( I_{12}(x\varpi_4 + y\varpi_3) \) denote the Harish–Chandra module of the normalized induced spherical degenerate principal series representation

\[
\text{Ind}_{M_{12} \cdot (\mathbb{R}^+)^2 \cdot N_{12}}^{F_4}(1_{M_{12}} \otimes a^{x\varpi_4 + y\varpi_3} \otimes 1).
\]

It has infinitesimal character \( x\varpi_3 + y\varpi_4 + \rho(\mathfrak{sl}_3) \).

**Lemma 8:** Suppose \( x \) and \( y \) satisfy Lemma 4(ii). Then the spherical degenerate principal series representation \( I_{12}(x\varpi_4 + y\varpi_3) \) is irreducible.

**Proof:** Suppose \( I_{12}(x\varpi_4 + y\varpi_3) \) is reducible. Hence it contains a non-spherical irreducible subquotient, say \( \pi'_1 \). Now \( \pi'_1 \) will also satisfy Lemma 7 and it will contain one of non-trivial minimal \( K_1 \)-types given in Table 2 or the last column of Table 1. It is straightforward to check that none of these non-trivial \( K_1 \)-types is a \( K_1 \)-type of the degenerate principal series representation.
Now we can state the main result for $E_8$. Let $\pi_1$ and $\pi_2$ be irreducible Harish–Chandra modules of $F_4$ and $G_2$ respectively.

**Theorem 9**: Suppose $\pi_1 \boxtimes \pi_2$ is a quotient of the minimal representation of $E_8$. Then:

(i) Suppose that $\pi_2$ is a generic spherical principal series representation with infinitesimal character $\lambda_2 = x\varpi_1 + y\varpi_2$. Then $\pi_1$ is the irreducible degenerate principal series representation $I_{12}(x\varpi_4 + y\varpi_3)$.

(ii) Suppose $\pi_1 = I_{12}(x\varpi_4 + y\varpi_3)$ such that Lemma 4(ii) holds for these $x$ and $y$. Then $\pi_2$ is a generic spherical principal series representation with infinitesimal character $\lambda_2$.

**Proof**: We will first prove (ii). By the correspondence of infinitesimal characters, the infinitesimal character of $\pi_2$ satisfies Lemma 4. Hence $\pi_2$ is a generic principal series representation. By Lemma 2, it cannot be the non-spherical representation and hence it is the spherical principal series. This proves (ii).

We will now prove (i). By the correspondence of infinitesimal characters, the infinitesimal character of $\pi_1$ satisfies Lemma 7. Hence $\pi_2$ contains one of the minimal $K_1$-types in Table 2 or the last column of Table 1. By Lemma 2, the minimal $K_1$-type must be the trivial $K_1$-type so $\pi_1$ is the unique spherical representation with infinitesimal character $x\varpi_4 + y\varpi_3 + \rho(\mathfrak{sl}_3)$.

Now $I_{12} := I_{12}(x\varpi_4 + y\varpi_3)$ is also an irreducible spherical representation with the same infinitesimal character as $\pi_1$. Hence $\pi_1 = I_{12}$ because irreducible spherical representations are uniquely determined by their infinitesimal characters. This completes the proof of Theorem 9.

### 3. Littlewood–Richardson (LR) Rule

The rest of this paper is devoted to the proof of Lemma 2. First we recall the famous Littlewood–Richardson branching rule (LR rule for short) for the restriction of representations from $\mathfrak{gl}_{n+m}$ to $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ which we will use many times later. Recall that a partition $\nu$ of $n$ parameterizes an irreducible representation $V_\nu$ of $\mathfrak{gl}_n$.

**Theorem 10** (Littlewood–Richardson rule): Let $\lambda$, $\mu$ and $\nu$ be a partition of $m+n$, $m$ and $n$, respectively. The multiplicity $c_{\mu\lambda\nu}$ of $V_\mu \otimes V_\nu$ in $V_\lambda$ is equal to the number of way the Young diagram for $\mu$ can be expanded to the Young diagram of $\lambda$ by a strict $\nu$-expansion. More precisely, if $\nu = (\nu_1, \ldots, \nu_k)$, a $\nu$-expansion is obtained by first adding $\mu_1$ boxes, with no two boxes in the same
column, and putting the integer 1 in each of these boxes. We then add \( \mu_2 \) boxes with a 2 in the same fashion, and so on. When integers in the boxes are listed from right to left, starting with the top row and working down, an expansion is called strict if for every \( t \) between 1 and \( \mu_1 + \cdots + \mu_k \) the first \( t \) integers on this list contain each integer \( z \) between 1 and \( k-1 \) at least as many times as the next integer \( z+1 \).

Remark: In order to calculate LR coefficients efficiently, we note the following two properties necessarily satisfied by any strict \( \nu \)-expansion:

- The integers in boxes are strictly increasing in each column, and are increasing (but not necessarily strictly) in each row.
- The first row can contain only boxes with a 1, the second row can only contain boxes with 1 and 2, and so on.

4. Proof of Lemma 2 (i) for \( E_7 \)

We will work exclusively with complexified Lie algebras.

Let \( \mathfrak{t}_2 = \mathfrak{sl}_{2,1} + \mathfrak{sl}_{2,s} \) be the Lie algebra of \( K_2 \) where \( \mathfrak{sl}_{2,1} \) corresponds to a long root, and \( \mathfrak{sl}_{2,s} \) to a short root. We will identify the Lie algebra \( \mathfrak{k} = \mathfrak{sl}_8 \) of \( K \) with the set of 8 by 8 traceless matrices. Then \( \mathfrak{sl}_{2,1} \subseteq \mathfrak{sl}_8 \) can be arranged to occupy upper-left \( 2 \times 2 \) block in \( \mathfrak{sl}_8 \). The centralizer of \( \mathfrak{sl}_{2,1} \) in \( \mathfrak{sl}_8 \) is \( \mathfrak{gl}_6 \) where the center of \( \mathfrak{gl}_6 \) consists of the diagonal matrices

\[
\{ d(z) := \text{diag}(-3z, -3z, z, z, z, z, z, z) : z \in \mathbb{C} \}.
\]

The identification with \( \mathfrak{gl}_6 \) is done so that the central elements \( d(z) \) act by \( z \) on the one dimensional representation with the highest weight \( (1, 0, 0, 0, 0, 0) \). It follows easily from LR rule that

\[
V_{n\mathfrak{k}}(n\mathfrak{w}_4)^{\mathfrak{sl}_{2,1}} = \sum_{k=0}^{n} V_{\mathfrak{gl}_6}(k, k, 2k-n, 2k-n, k-n, k-n).
\]

Next, using \( \mathbb{C}^6 = \mathbb{C}^2 \otimes \mathbb{C}^3 \) we can embed \( \mathfrak{sl}_{2,s} + \mathfrak{gl}_3 \) into \( \mathfrak{gl}_6 \). In this way we have completely described an embedding of

\[
\mathfrak{k}_1 + \mathfrak{k}_2 = \mathfrak{gl}_3 + (\mathfrak{sl}_{2,1} + \mathfrak{sl}_{2,s})
\]

into \( \mathfrak{k} \). In order to prove Lemma 2, we need to analyze the \( \mathfrak{gl}_3 \)-invariants of representations appearing on the right hand side of (5). First of all, if the center of \( \mathfrak{gl}_3 \) acts trivially, then \( n = 2k \) in (5). We will now denote

\[
V_k := V_{\mathfrak{gl}_6}(k, k, 0, 0, -k, -k).
\]
Next, we consider 
\[ \mathfrak{t}_1 = \mathfrak{gl}_3 \subset \mathfrak{gl}_3 + \mathfrak{gl}_3 \subset \mathfrak{gl}_6 \]
where \( \mathfrak{gl}_3 \) is embedded diagonally. Suppose \( W \otimes W' \) is a representation of \( \mathfrak{gl}_3 + \mathfrak{gl}_3 \) appearing in the restriction of \( V_k \). Then \( W \otimes W' \) gives rise to a one dimensional invariant subspace of \( f_1 \) if and only if \( W' = W^* \), the dual representation of \( W \). If \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) is the highest weight of \( W \), then \( (W \otimes W^*)^\mathfrak{t}_1 \) is contained in the weight \( 2 \sum_1 \lambda_i \) space for \( \mathfrak{sl}_2, s \).

In order to prove Lemma 2(i), we need to show that the representation \( S^2(\mathbb{C}^2) \) of \( \mathfrak{sl}_2, s \) does not appear in \( V_k^{\mathfrak{t}_1} \). To that end, it suffices to show that the dimensions of the weight 2 space and the weight \((-4)\) space for \( \mathfrak{sl}_2, s \) are equal. This follows immediately from the following proposition.

**Proposition 11:** Let \( W \) be a representation of \( \mathfrak{gl}_3 \) of highest weight \( \lambda \), and let \( m(k, \lambda) \) denote the multiplicity of \( W \otimes W^* \) in \( V_k \). If \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) with \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \), then \( m(\lambda, k) = m(\lambda^-, k) \) where \( \lambda^- = (\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1) \).

**Proof:** We need to calculate \( m(\lambda) \). This is accomplished by using LR rule in the following lemma.

**Lemma 12:** Suppose \( W = V_{\mathfrak{gl}_3}(\lambda_1, \lambda_2, \lambda_3) \). Assume first that \( \lambda_2 \leq 0 \). Then the multiplicity of \( W \otimes W^* \) in \( V_k \) is zero unless \( k \leq \lambda_1 + \lambda_2 \). If it is not zero then it is equal to

\[
\begin{cases}
\min(\lambda_2 - \lambda_1, \lambda_1) + 1 & \text{if } k \geq \lambda_1 - \lambda_3 \\
\min(\lambda_2 - \lambda_1 + k, \lambda_3 + k) + 1 & \text{if } k < \lambda_1 - \lambda_3.
\end{cases}
\]

To find the multiplicity when \( \lambda_2 \geq 0 \), we use the symmetry by interchanging \( W \) with \( W^* \). In this case the multiplicity is

\[
\begin{cases}
\min(\lambda_1 - \lambda_2, -\lambda_3) + 1 & \text{if } k \geq \lambda_1 - \lambda_3 \\
\min(\lambda_3 - \lambda_2 + k, k - \lambda_1) + 1 & \text{if } k < \lambda_1 - \lambda_3.
\end{cases}
\]

**Proof:** In order apply the LR rule, we consider \( U = \det^k \otimes W, U' = \det^k \otimes W^* \) and \( E = \det^k \otimes V_k \). Hence \( U \) has the highest weight \((a,b,c) = (k,k,k) + \lambda, U'\) has the highest weight \((a',b',c') = (k,k,k) + \lambda^*\) and \( E \) has the highest weight \((2k,2k,k,k,0,0)\). The multiplicity of \( U \otimes U' \) in \( E \) is equal to \( m(k,\lambda) \).

From the LR rule, \( 0 \leq c \leq k \) and \( 0 \leq b \leq a \leq 2k \). The same is true for \( a',b',c' \). Since \( b + b' = 2k \), by interchanging the role of \( U \) and \( U' \), we may assume that \( b \leq k \leq b' \).

Let \( Y, Y' \) and \( Z \) denote the Young diagrams of \( U, U' \) and \( E \) respectively. We place \( Y \) inside \( Z \) and fill in the remaining spaces in \( Z \) with the boxes from \( U' \).
By the remark after Theorem 10 there is a unique way of filling the first 2 rows of $Z$, namely, $c'$ 1-boxes on the first row and, $a - b = b' - c'$ 1-boxes on the second row and $c'$ 2-boxes on the second row. All the 3-boxes (there are $c'$ of them) appear on the 4-th row of $Z$. The Young diagram is given in the figure below.

![Young diagram](image)

It shows that $c' \geq b' - k = \lambda_2$.

It remains to fill the last 2 rows of $Z$ with 1-boxes and 2 boxes. Suppose $c + c' < k$ as shown in the Young diagram above, then some of the boxes are uniquely determined as shown in the shaded area. It remains to fill $x$ with 1-boxes and 2-boxes. Let

$$z = \max(0, k - c - c') = \max(0, a - c - k).$$

Now $x$ has $b - c - z$ boxes which is the same number as the remaining 1-boxes to be filled. Hence the number of ways of filling is equal to the number of ways of putting 1-boxes into $y = \min(c, k - c')$. This is equal to

$$\min(b - c - z, c, k - c') + 1 = \min(b - c - z, c, a - k) + 1$$
$$= \min(b - c, b - a + k, c, a - k) + 1.$$

If we substitute $\lambda_i$ back into $a, b, c$ above, and the condition $k \geq \lambda_1 - \lambda_3$ (or $k < \lambda_1 - \lambda_3$), we would recover the multiplicity stated in the lemma. The lemma is proved.

We will now prove Proposition 11. We first assume that $\lambda_2 \leq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, or $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = -2$. One easily checks that the multiplicity is

$$\begin{cases} 
\lambda_2 - \lambda_3 + 1 & \text{if } k \geq \lambda_1 - \lambda_3 \\
\lambda_2 - \lambda_1 + k + 1 & \text{if } k < \lambda_1 - \lambda_3.
\end{cases}$$

Assume now that $\lambda_2 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, or $\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = -2$. 


One easily checks that the multiplicity is

\[
\begin{cases}
\lambda_1 - \lambda_2 + 1 & \text{if } k \geq \lambda_1 - \lambda_3 \\
\lambda_3 - \lambda_2 + k + 1 & \text{if } k < \lambda_1 - \lambda_3.
\end{cases}
\]

Proposition 11 follows. \( \square \)

5. Proof of Lemma 2(ii) for \( E_7 \)

Recall that in Section 4 we defined the following sequence of embeddings:

\[
\mathfrak{t}_1 = \mathfrak{gl}_3 \to \mathfrak{gl}_3 \times \mathfrak{gl}_3 \to \mathfrak{gl}_6 \to \mathfrak{sl}_8 = \mathfrak{k}.
\]

Hence we get an embedding of the complex groups \( \text{GL}_3 \) into \( \text{SL}_8 \). We will denote this \( \text{GL}_3 \) by \( \text{GL}_c^3 \). Since \( K(C) = \text{SL}_8/\mu_2 \) this gives an embedding of \( \text{GL}_c^3/\mu_2 \) into \( K(C) \). The group \( \text{GL}_c^3/\mu_2 \) can be identified with \( K_1(C) = \text{GL}_3 \) as follows. Let \( zs \in \text{GL}_c^3 \) where \( z \) is a scalar matrix and \( s \in \text{SL}_3 \). Then

\[
\phi(zs) = z^{-2}s
\]

defines a map from \( \text{GL}_c^3 \) onto \( \text{GL}_3 \) with kernel \( \mu_2 \). When pulled back by \( \phi \) the minimal \( K_1(C) \)-type with the highest weight representation \( (1, 1, 0) \) becomes the representation \( V_{\mathfrak{k}_1}(-1, -1, -2) \) of \( \text{GL}_c^3 \).

5.1. We will begin the proof of Lemma 2(ii) which states that the subrepresentation \( V_{\mathfrak{k}_1}(-1, -1, -2) \) does not occur in the space of \( \mathfrak{sl}_{2, s} \)-invariants on right hand side of (5). First of all, notice that not all summands in (5) will contain \( V_{\mathfrak{k}_1}(-1, -1, -2) \). Indeed the central character of \( V_{\mathfrak{k}_1}(-1, -1, -2) \) is \(-4\). This implies that \( 4(-n + 2k) = -4 \) which is equivalent to \( 2k = n + 1 \). Thus, we only need to consider

\[
V_{\mathfrak{k}_1}' := V_{\mathfrak{gl}_6}(k, k, -1, -1, -k - 1, -k - 1).
\]

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) and \( \lambda' = (\lambda_1', \lambda_2', \lambda_3') \) denote two highest weights of \( \mathfrak{gl}_3 \). We would like to know the multiplicities of

\[
V_{\mathfrak{k}_1}(-1, -1, -2) \subset V_{\mathfrak{gl}_6}(\lambda) \otimes V_{\mathfrak{gl}_6}(\lambda') \subset V_{\mathfrak{k}_1}'
\]

such that \( S = \sum_i \lambda_i - \lambda_i' \in \{0, 2\} \). For technical reason, we allow \( S = -2 \) as well. Note that \( S \) is the weight for the torus of \( \mathfrak{sl}_{2, s} \) acting on \( W \otimes W' \). Since the central character, with respect to \( \mathfrak{t}_1 \), is \(-4 = \sum_i \lambda_i + \lambda_i' \), we can rewrite

\[
S = \sum_i \lambda_i - \lambda_i' = \sum_i 2\lambda_i - (\lambda_i + \lambda_i') = 2\left(\sum_i \lambda_i + 2\right).
\]
Hence $S \in \{-2, 0, 2\}$ is equivalent to $\sum_i \lambda_i \in \{-3, -2, -1\}$.

We consider the first inclusion in (7).

**Lemma 13:** The multiplicity of $V_{\ell i}((-1, -1, -2))$ in

$$V_{\ell i}((\lambda_1, \lambda_2, \lambda_3)) \otimes V_{\ell i}((\lambda'_1, \lambda'_2, \lambda'_3))$$

is either 0 or 1. It is 1 if and only if one of the following situations holds:

(i) $\lambda_1 + \lambda'_3 = -2, \lambda_2 + \lambda'_2 = \lambda_3 + \lambda'_1 = -1$,
(ii) $\lambda_1 > \lambda_2, \lambda_1 + \lambda'_3 = -1, \lambda_2 + \lambda'_2 = -2, \lambda_3 + \lambda'_1 = -1$,
(iii) $\lambda_2 > \lambda_3, \lambda_1 + \lambda'_3 = \lambda_2 + \lambda'_2 = -1, \lambda_3 + \lambda'_1 = -2$.

Note that (iii) is obtained from (i) by interchanging the role of $\lambda$ and $\lambda'$. The proof is just another exercise in LR rule and we will leave it to the reader.

We will see later in the proof of Lemma 14 that in order for $V_k'$ to contain $V_{\ell i}((\lambda_1, \lambda_2, \lambda_3)) \otimes V_{\ell i}((\lambda'_1, \lambda'_2, \lambda'_3))$ it is necessary that we have

$$k \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq -k - 1, \quad k \geq \lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq -k - 1.$$

From now on, we will refer to the three cases in Lemma 13 satisfying (8) as Cases (i), (ii) and (iii) respectively. For Case (i), one can show that $\lambda_1 \leq k - 1$

Let $m(\lambda, \lambda', k)$ denote the multiplicities of $V_{\ell i}((\lambda_1, \lambda_2, \lambda_3)) \otimes V_{\ell i}((\lambda'_1, \lambda'_2, \lambda'_3))$ in $V_k'$ such that tensor product also contains $V_{\ell i}((-1, -1, -2))$.

**Lemma 14:** Suppose $S = \sum_i \lambda_i - \lambda'_i = 0$ or 2. Then the multiplicities $m(\lambda, \lambda', k)$ of the three cases in Lemma 13 are given in the table below.

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>$k \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq -k - 1$</th>
<th>$k \geq \lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq -k - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>if $k \leq \lambda_1 - \lambda_3$</td>
<td>if $\lambda_2 \geq 0$</td>
<td>$k - \lambda_2 + \lambda_3 + 1$</td>
</tr>
<tr>
<td>if $k &gt; \lambda_1 - \lambda_3$</td>
<td>if $\lambda_2 \leq -1$</td>
<td>$k - \lambda_1 + \lambda_2 + 1$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \geq 0$</td>
<td>$\lambda_1 - \lambda_2 + 1$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \leq -1$</td>
<td>$\lambda_2 - \lambda_3 + 1$</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>if $k \leq -2 + \lambda_1 - \lambda_2$</td>
<td>$k - \lambda_2 + \lambda_3 + 1$</td>
</tr>
<tr>
<td>if $k &gt; \lambda_1 - \lambda_3 - 1$</td>
<td>if $\lambda_2 \geq 0$</td>
<td>$k - \lambda_1 + \lambda_2 + 2$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \leq -1$</td>
<td>$\lambda_1 - \lambda_2$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \geq 0$</td>
<td>$\lambda_2 - \lambda_3 + 1$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \leq -1$</td>
<td>$\lambda_2 - \lambda_3 + 1$</td>
</tr>
<tr>
<td>Case (iii)</td>
<td>if $k \leq \lambda_1 - \lambda_3 - 1$</td>
<td>$k - \lambda_2 + \lambda_3 + 2$</td>
</tr>
<tr>
<td>if $k &gt; \lambda_1 - \lambda_3 - 1$</td>
<td>if $\lambda_2 \geq 0$</td>
<td>$k - \lambda_1 + \lambda_2 + 1$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \leq -1$</td>
<td>$\lambda_1 - \lambda_2 + 1$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \geq 0$</td>
<td>$\lambda_2 - \lambda_3$</td>
</tr>
<tr>
<td></td>
<td>if $\lambda_2 \leq -1$</td>
<td>$\lambda_2 - \lambda_3$</td>
</tr>
</tbody>
</table>

**Proof:** We would like to apply the LR rule so we set:

$$U = \det \otimes V_{\ell i}((\lambda_1, \lambda_2, \lambda_3)), \quad U' = \det \otimes V_{\ell i}((\lambda'_1, \lambda'_2, \lambda'_3))$$
and

\[ E' = \det \otimes V'_k. \]

The highest weight of \( U \) is \((a, b, c) = (k+1, k+1, k+1) + \lambda\), the highest weight of \( U' \) is \((a', b', c') = (k+1, k+1, k+1) + \lambda'\), and the highest weight of \( E' \) is \((2k+1, 2k+1, k, k, 0, 0)\). Then

\[ V_{t_1}(2k+1, 2k+1, 2k) \subset U \otimes U' \subset E'. \]

and the multiplicities are not affected. By interchanging \( U \) and \( U' \) if necessary, we may assume that \( b \leq k \), that is, \( \lambda_2 \leq -1 \). Let \( Y, Y' \) and \( Z \) be the Young diagrams of \( U, U' \) and \( E' \). We embed \( Y \) into \( Z \) and we fill \( Z \) with the boxes of \( Y' \). An almost identical argument as before gives the same figure as Figure 1. Here \( z = \max(0, k - c - c') \). The number of 2-boxes in the second row is not less than the number \( c' \) of 3-boxes.

In all cases the number of remaining 1-boxes is not greater than the number of boxes in \( x \). There are \( b - c - z - e \) remaining 1-boxes; where \( e = 0 \) in Cases (i) and (ii) and \( e = 1 \) in Case (iii). Hence the multiplicity of \( U \otimes U' \) in \( E' \) is equal to the number of ways of filling the remaining 1-boxes in \( y = \min(c, k - c') \) which is

\[ \min(c, k - c', b - c - z - e) + 1 = \min(c, k - c', b - c - e, b - k + c' - c) + 1. \]

More explicitly, the multiplicity in the three cases is:

- Case (i): \( \min(c, a - k, b - c, k - a + b) + 1 \)
  \[ = \min(\lambda_3 + k + 1, \lambda_1 + 1, \lambda_2 - \lambda_3, k - \lambda_1 + \lambda_2) + 1 \]
- Case (ii): \( \min(c, a - k - 1, b - c, k + 1 - a + b) + 1 \)
  \[ = \min(\lambda_3 + k + 1, \lambda_1, \lambda_2 - \lambda_3, k + 1 - \lambda_1 + \lambda_2) + 1 \]
- Case (iii): \( \min(c, a - k - 1, b - c - 1, k - a + b) + 1 \)
  \[ = \min(\lambda_3 + k + 1, \lambda_1, \lambda_2 - \lambda_3 - 1, k - \lambda_1 + \lambda_2) + 1. \]

The fact that \( \sum \lambda_i \in \{-3, -2, -1\} \) and \( \lambda_2 \leq -1 \) implies that \( \lambda_1 + \lambda_3 + 1 \geq \lambda_2 \). Then the above multiplicity simplifies to

- Case (i): \( \min(\lambda_2 - \lambda_3, k - \lambda_1 + \lambda_2) + 1 \)
- Case (ii): \( \min(\lambda_2 - \lambda_3, k + 1 - \lambda_1 + \lambda_2) + 1 \)
- Case (iii): \( \min(\lambda_2 - \lambda_3 - 1, k - \lambda_1 + \lambda_2) + 1. \)

Here Case (ii)' refers to Case (ii) except the situation where \( \sum \lambda_i = -3 \) and \( \lambda_2 = -1 \), which is of no use to us.
By interchange the role of $\lambda$ and $\lambda'$, we obtain the cases where $\lambda_2 \geq 0$. This is where we need the fact that $\sum_i \lambda = -3$ so that $\sum_i \lambda'_i = -1$.

The table in Lemma 14 follows immediately by comparing the calculations made above. This proves Lemma 14.

Proof of Lemma 2(ii) for $E_7$: In order to prove the lemma, it suffices to show that the dimensions of the weight 2 space and weight 0 space for $\mathfrak{sl}_{2,s}$ are equal. This is equivalent to

$$(9) \quad \sum m(\lambda, \lambda', k) = \sum m(\lambda, \lambda', k)$$

where the first (resp. second) sum is taken over all $(\lambda, \lambda')$ satisfying Lemma 13 and such that $S = \sum_i \lambda_i - \lambda'_i = 2$ (resp. $S = 0$). We have seen before that $S = 2$ (resp. $S = 0$) is equivalent to $\sum_i \lambda_i = -1$ (resp. $\sum_i \lambda_i = -2$).

We refer to Lemma 14. Suppose $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfies $\sum_i \lambda = -1$. We define

$\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = (-\lambda_3 - 1, -\lambda_2 - 1, -\lambda_1 - 1)$.

We gather some properties of the transformation $\lambda \mapsto \tilde{\lambda}$.

(a) Since $\sum \lambda = -1$, we have $\sum \tilde{\lambda} = -2$.

(b) We have $\lambda_1 - \lambda_3 = \lambda_1 - \lambda_3, \lambda_1 - \lambda_2 = \tilde{\lambda}_2 - \tilde{\lambda}_3$. $\lambda_2 - \lambda_3 = \tilde{\lambda}_1 - \tilde{\lambda}_2$.

(c) If $\lambda_2 \geq 0$, then $\lambda_2 \leq -1$. Conversely if $\lambda_2 \leq -1$, then $\lambda_2 \geq 0$.

(d) If $\lambda$ belongs to Case (i) (resp. Case (ii), Case (iii)), then $\tilde{\lambda}$ also belongs to Case (i) (resp. Case (iii), Case (ii)).

We will say more about (d). Indeed if we refer to the table in Lemma 14, the transformation $\lambda \mapsto \tilde{\lambda}$ preserves multiplicities. For example, if we refer to the table in Lemma 14, then the transformation sends the first (resp. second) line of Case (i) for $(\lambda_1, \lambda_2, \lambda_3)$ to the second (resp. first) line of Case (i) for $(\lambda_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$.

This proves (9) and Lemma 2(ii).

6. Proof of Lemma 2(i) for $E_8$

6.1. Lie subalgebras of $\mathfrak{e}_8$. Let

$\mathfrak{so}_{16}, \mathfrak{fraksl}_{2,l}^{F} + \mathfrak{sp}_6$ and $\mathfrak{sl}_{2,l}^{G} + \mathfrak{sl}_{2,s}$

be the complexified Lie algebras of the maximal compact subgroups $K, K_1, K_2$ of $E_{8,8}$, $F_{4,4}$ and $G_{2,2}$ respectively. The Lie algebra $\mathfrak{so}_{16}$ contains

$\mathfrak{so}_{12} + \mathfrak{so}_4 = \mathfrak{so}_{12} + (\mathfrak{sl}_{2,l}^{F} + \mathfrak{sl}_{2,l}^{G})$. 

The standard representation $\mathbb{C}^{12}$ of $\mathfrak{so}_{12}$ can be written as a product of two symplectic spaces $\mathbb{C}^6 \otimes \mathbb{C}^2$. This gives an embedding of $\mathfrak{sp}_6 + \mathfrak{sl}_2$ into $\mathfrak{so}_{12}$, and the embedding of $\mathfrak{t}_1 + \mathfrak{t}_2$ into $\mathfrak{t}$ is described completely. Using the notation of the root system of $\mathfrak{e}_8$ in [1], $\mathfrak{sl}_{G,2}^F$ and $\mathfrak{sl}_{G,2}^G$ correspond to the simple roots $\varepsilon_1 + \varepsilon_2$ and $\varepsilon_2 - \varepsilon_1$ respectively.

We need one additional subalgebra of $\mathfrak{so}_{12}$. If we decompose the standard representation $\mathbb{C}^{12} = \mathbb{C}^6 \oplus (\mathbb{C}^6)^*$ of $\mathfrak{so}_{12}$ into two isotropic subspaces, then we get $\mathfrak{sp}_6 \subset \mathfrak{gl}_6 \subset \mathfrak{so}_{12}$. The center $\mathfrak{gl}_1$ of $\mathfrak{gl}_6$ forms a torus of $\mathfrak{sl}_{2, \mathbb{C}}$.

Recall that $\mathfrak{so}_{16}$ has two maximal parabolic subalgebras with Levi component isomorphic to $\mathfrak{gl}_8$. We set $\mathfrak{gl}_{G,2}^G$ and $\mathfrak{gl}_{G,2}^F$ to be the Lie subalgebras such that $\mathfrak{so}_{16} \supset \mathfrak{gl}_{G,2}^G \supset \mathfrak{gl}_6 + \mathfrak{gl}_{G,2}^F = \mathfrak{gl}_6 + (\mathfrak{gl}_1 + \mathfrak{sl}_{G,2}^F)$.

Similarly we define $\mathfrak{gl}_{F,2}^G$ and $\mathfrak{gl}_{F,2}^F$. Note that the center of $\mathfrak{gl}_{G,2}^F$ is equal to the split torus of $\mathfrak{sl}_{F,2}$, and the center of of $\mathfrak{gl}_{F,2}^G$ is equal to the split torus of $\mathfrak{sl}_{G,2}$.

**Two branching rules.** We will state two branching rules that we will need later. The first branching rule is a special case of one due to T. Enright and M. Hunziker. One can also give a direct proof using Borel–Weil theorem.

**Lemma 15:** Let $\varpi_8$ denote the fundamental weight corresponding to the half-spin representation of $\mathfrak{so}_{16}$ acting on $\mathfrak{p}$. Then $\varpi_8$ is perpendicular to the roots of $\mathfrak{gl}_{F,2}^F$, and

(i) \[ V_{\mathfrak{so}_{16}}(n\varpi_8)|_{\mathfrak{gl}_{F,2}^F} = \sum V_{\mathfrak{gl}_6}(a_1, a_2, a_3, a_4, a_5, a_6) \]

where the sum is taken over $n/2 \geq a_1 \geq \cdots \geq a_4 \geq -n/2$ such that $a_i - n/2 \in \mathbb{Z}$, and

(ii) \[ V_{\mathfrak{so}_{16}}(n\varpi_8)|_{\mathfrak{gl}_{G,2}^G} = \sum V_{\mathfrak{gl}_6}(a_1, a_2, a_3, a_4, a_5, a_6) \]

where the sum is taken over $n/2 \geq a_1 \geq \cdots \geq a_5 \geq -n/2$ such that $a_i - n/2 \in \mathbb{Z}$.

We now state the second branching rule.

**Lemma 16:** Consider $\mathfrak{sp}_6$ in $\mathfrak{gl}_6$. Let $\lambda = (\lambda_1, \ldots, \lambda_6)$ be a highest weight of $\mathfrak{gl}_6$.

(i) The dimension of $(V_{\mathfrak{gl}_6}(\lambda))^{\mathfrak{sp}_6}$ is either 0 or 1. It is 1 if and only if $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, and $\lambda_5 = \lambda_6$. 
(ii) The representation $V_{\mathfrak{gl}}(\lambda)$ contains the representation $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$ with either multiplicity 0 or 1. It is 1 if and only if

$$r = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6.$$ 

Proof: Part (i) follows from the Cartan–Helgason theorem (see page 535 in [6]). For (ii), we consider

$$V_{\mathfrak{gl}}(\lambda) \otimes S^r(\mathbb{C}^6) = \sum_{\lambda'} c_{\lambda',r}^{\lambda} V_{\mathfrak{gl}}(\lambda')$$

as representations of $\mathfrak{gl}_6$. Here the LR number $c_{\lambda',r}^{\lambda}$ is either 0 or 1. The representation $V_{\mathfrak{gl}}(\lambda)$ contains $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$ if and only if some $V_{\mathfrak{gl}}(\lambda')$ on the right hand side of the equation contains the trivial representation of $\mathfrak{sp}_6$. Now (ii) follows from (i).

**Lemma 17:** Suppose $V_{\mathfrak{so}}(n\varpi_8)$ contains the irreducible representation $S^s(\mathbb{C}^2) \boxtimes S^r(\mathbb{C}^6)$ of $\mathfrak{sl}_2^s + \mathfrak{sp}_6$, then $r \geq s$ and $r \equiv s \pmod{2}$.

Proof: If $V_{\mathfrak{gl}}(\lambda)$ contains $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$, then by Lemma 16(i),

$$r = \sum_{i=1}^{3} \lambda_{2i-1} - \lambda_{2i}.$$ 

By Lemma 15(ii), it is enough to check if $S^s(\mathbb{C}^2) \boxtimes V_{\mathfrak{gl}}(\lambda)$ is a submodule of $V_{\mathfrak{gl}}(a, a, b, b, c, c, d, d)$. In other words, we need to find the values of $s$ such that the LR number $c_{(s+f,f)}^{(a,a,b,b,c,c,d,d),\lambda} \neq 0$ where $f$ is arbitrary. The lemma follows from a direct calculation.

**Proposition 18:** The generic principal series representations of $\mathbb{F}_{4,4}$ are not quotients of the minimal representations of $\mathfrak{E}_{8,8}$.

Proof: Indeed there are three families of generic principal series representations and they contain the $K_1$-types $S^4(\mathbb{C}^2) \boxtimes \mathbb{C}$, $S^2(\mathbb{C}^2) \otimes \mathbb{C}$ and $S^5(\mathbb{C}^2) \boxtimes \mathbb{C}^6$ respectively. On the other by Lemma 17 these $K_1$-types do not appear in the the restriction of the minimal representation.

We will begin the proof of Lemma 2(i). We need to show that the representation $S^2(\mathbb{C}^2)$ of $\mathfrak{sl}_2^s$ does not appear in $(V_{\mathfrak{so}}(n\varpi_8))^{\mathfrak{sl}_2^s + \mathfrak{sl}_2^s + \mathfrak{sp}_6}$. First of all, by the standard branching rules for $\mathfrak{so}_{16} \downarrow \mathfrak{so}_{15} \downarrow \cdots \downarrow \mathfrak{so}_4 = \mathfrak{sl}_2^s + \mathfrak{sl}_2^s,$
the space of invariants \((V_{\sigma_1}(n\varpi_8))^{st_{2,i}+st_{2,j}}\) is zero if \(n\) is odd. Hence we will assume that \(n\) is even. Next, by Lemma 17,
\[
(V_{\sigma_1}(n\varpi_8))^{st_{2,i}+st_{2,j}+sp_\theta} = (V_{\sigma_1}(n\varpi_8))^{sp_\theta}.
\]
Here \(gl_2^C = gl_1 + sl_2^C\) where \(gl_1\) is the torus of \(sl_2^C\). By Lemma 16(i) \(V_{gl_6}(\lambda)\) contains \(1_{sp_\theta}\) if and only if \(\lambda = (a, a, b, b, c, c)\). For such a \(\lambda\), we have
\[
1_{sp_\theta} \cong 1_{gl_2^C} \subset V_{gl_6}(\lambda) \cong 1_{gl_2^C} \subset V_{gl_6}(a_1, a_2, a_3, a_4, a_5) \subset V_{\sigma_1}(n\varpi_8).
\]
Here \(a, b, c, a_i \in \mathbb{Z}\). The first and the last containments are of multiplicity one due to Lemmas 16(i) and 15(ii) respectively.

The center \(gl_1\) of \(gl_6\) is the torus of \(sl_{2,s}\) and it acts on \(V_{gl_6}(\lambda)\) by \(\sum_i \lambda_i\). We will show that the representation \(S^2(\mathbb{C}^2)\) of \(sl_{2,s}\) does not occur by showing that the dimensions of the weight 4 space and the weight \((-2)\) space of \(sl_{2,s}\) are the same on \(V_{\sigma_1}(n\varpi_8)^{st_{2,i}+st_{2,j}+sp_\theta}\). Now, \(V_{gl_6}(\lambda)\) contributes to the weight 4 or -2 if \(2a + 2b + 2c = 4\) or -2, respectively. Thus, in order to obtain the desired result it suffices to show the following.

**Proposition 19:** Let \(\lambda = (a, a, b, b, c, c)\) be such that \(a + b + c = 2\) and let \(m(a, b, c; a_1, a_2, \ldots, a_5)\) be the multiplicity of the middle inclusion in (10). Then
\[
m(a, b, c; a_1, a_2, \ldots, a_5) = m(a-1, b-1, c-1; a_1, a_2, \ldots, a_5).
\]

**Proof:** In order to verify the proposition we need to calculate the multiplicities. This will be accomplished using the Littlewood–Richardson rule in the following lemma.

**Lemma 20:** Suppose \(a + b + c = \pm 1, \pm 2\). Let \(m = m(a, b, c; a_1, a_2, \ldots, a_5)\) be the multiplicity of the middle inclusion in (10). If \(m \neq 0\), then it is necessary that \(a_1 \geq a \geq b \geq c \geq a_1\) and
\[
\lambda_2 = a_2, \lambda_4 = a_3, \lambda_6 = a_4, a_5 = -a_1.
\]

Furthermore, the table below shows the values of \(m\) for different values of \(a, b, c, a_1, a_2\) under the above conditions.

| \(b \geq 0\) | \(a - c \leq a_1\) | \(a - b + 1\) |
| \(a - c \geq a_1\) | \(a_1 - b + c + 1\) |
| \(b \leq 0\) | \(a - c \leq a_1\) | \(b - c + 1\) |
| \(a - c \geq a_1\) | \(a_1 - a + b + 1\) |

If \(m\) takes negative values in the above table, then we set \(m = 0\).

**Proof:** In order to apply the LR rule we twist representations with \(\det^{n/2}\). Let \(a'_i = a_i + n/2, a' = a + n/2, b' = b + n/2, c' = c + n/2\). First we place the Young
diagram $Y$ of $V_{\mathfrak{g}_b}(a', a', b', b', c', c')$ into that of $V_{\mathfrak{g}_a}(a_1', a_2', a_3', \ldots, a_5')$. Next we will fill the remaining spaces with $n/2$ copies of 1-boxes and 2-boxes. We show the Young diagrams below:

Without going into the details, LR rule shows that $a = a_2$, $b = a_3$, $c = a_4$, $a_5 = -a_1$. It remains to fill the shaded region $x$ and $y$ with $n/2 - c' = -c$ 2-boxes.

First suppose $b \geq 1$. Since $a + b + c = \pm 1, \pm 2$, we have

$$\text{Length of } y \geq -c \geq \text{Length of } x$$

By the LR rule, the multiplicity $m$ is

$$\min(a' - b', a_1' - a', a_1' - b' + c) + 1 = \min(a - b + 1, a_1 - a + 1, a_1 - b + c + 1).$$

Since $a + b + c \leq 2$, $a_1 - a + 1 \geq a_1 - b + c$. This proves the case $b \geq 1$.

Next by sending $(a, b, c) \mapsto (-c, -b, -a)$ and $a_i \mapsto a_{-i+1}$, we pass from the case $b \geq 1$ to $b \leq -1$.

The proof of the case $b = 0$ is similar and easier. The lemma is proved.

The proof of proposition is now identical to the proof of Proposition 11. We leave details to the reader.

7. Proof of Lemma 2(ii) for $E_8$

We continue with the notations in §6.1. The proof is almost identical with the one in §6 but more tedious. We need show that the representations $\mathbb{C}^2 \boxtimes \mathbb{C}^6$ and $S^2(\mathbb{C}^2) \boxtimes 1_{\mathfrak{sp}_6}$ of $\mathfrak{sl}_{2,1} + \mathfrak{sp}_6$ do not appear in $V_{\mathfrak{so}_1}(n \varpi_8)\mathfrak{sl}_{2,1} + \mathfrak{sl}_{2,2}$. The statement for $S^2(\mathbb{C}^2) \boxtimes 1_{\mathfrak{sp}_6}$ follows immediately from Lemma 17. We are left now with $\mathbb{C}^2 \boxtimes \mathbb{C}^6$. 
Once again, by the standard branching rules for $\mathfrak{so}_4 \downarrow \mathfrak{so}_3 \downarrow \cdots \downarrow \mathfrak{so}_2 = \mathfrak{sl}_2$ + $\mathfrak{sl}_2$, $\text{Hom}_{\mathfrak{sl}_2} \mathfrak{sl}_2^\mathbb{C} \otimes (\mathbb{C}^2, V_{\mathfrak{sl}_6}(n\varpi_8))$ is zero unless $n$ is odd. Hence we will assume that $n$ is odd. Next, by Lemma 17,

$$\text{Hom}_{\mathfrak{sp}_6 + \mathfrak{sl}_2} \mathfrak{sp}_6^\mathbb{C} \otimes \mathbb{C}^2, V_{\mathfrak{so}_6}(n\varpi_8)) = \text{Hom}_{\mathfrak{sp}_6 + \mathfrak{gl}_2} \mathfrak{sp}_6^\mathbb{C} \otimes \mathbb{C}^2, V_{\mathfrak{so}_6}(n\varpi_8))$$

where $\mathbb{C}_{-1/2} = V_{\mathfrak{gl}_6}(-1/2, -1/2)$ is a one-dimensional representation of $\mathfrak{gl}_2^G$. We recall Lemma 16(ii) that $V_{\mathfrak{gl}_6}(\lambda)$ contains $\mathbb{C}^6$ of $\mathfrak{sp}_6$ if and only if $\lambda$ is of the form

$$\lambda_I = (a + 1/2, a - 1/2, b, b, c, c),$$
$$\lambda_{II} = (a, a, b + 1/2, b - 1/2, c, c) \text{ or}$$
$$\lambda_{III} = (a, a, b, b, c + 1/2, c - 1/2).$$

We have inserted the ‘1/2’s so that there is more symmetry in our calculations. For $\lambda = \lambda_I, \lambda_{II}$ or $\lambda_{III}$, we have by Lemma 15(ii),

$$\mathbb{C}^6 \otimes \mathbb{C}_{-1/2} \subset V_{\mathfrak{gl}_6}(\lambda) \otimes \mathbb{C}_{-1/2} \subset V_{\mathfrak{gl}_6}(a_1, a_2, a_3, a_4, a_5) \subset V_{\mathfrak{so}_6}(n\varpi_8),$$

where $\lambda$ and $a_i$ lie in $\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$. The first and the last containments are of multiplicity one due to Lemma 16(ii) and 15(ii) respectively.

The center $\mathfrak{gl}_1$ of $\mathfrak{gl}_6$ is the torus of $\mathfrak{sl}_2$, and it acts on $V_{\mathfrak{gl}_6}(\lambda)$ by $\sum_i \lambda_i$. We will prove that the trivial representation of $\mathfrak{sl}_2$ does not occur, by showing that the dimensions of the 0-eigenspace and the 2-eigenspace of $\mathfrak{gl}_1$ are the same. The action of $\mathfrak{gl}_1$ implies that $2a + 2b + 2c = 0$ or $2$, that is, $a + b + c = 0$ or $1$.

Let $m = m(a, b, c; a_1, a_2, \ldots, a_5)$ denote multiplicity of the middle inclusion in (12). There is no ambiguity as to whether we are using $\lambda = \lambda_I, \lambda_{II}$ or $\lambda_{III}$ in the definition of $m$. This is because for $\lambda_I$ (resp. $\lambda_{II}, \lambda_{III}$), the entry $a$ (resp. $b$, $c$) is an integer while the rest of the entries are odd multiplies of 1/2.

**Lemma 21:** Let $\lambda = \lambda_I, \lambda_{II}$ or $\lambda_{III}$ and $m = m(a, b, c; a_1, a_2, \ldots, a_5)$ as above. Suppose $a + b + c = -1, 0$ or $1$. If $m \neq 0$, then it is necessary that all the entries in $\lambda$ lie in the closed interval $[-a_1, a_1]$ and

$$\lambda_2 = a_2, \lambda_4 = a_3, \lambda_6 = a_4, a_5 = -a_1.$$
Furthermore

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( b ) ( \geq \frac{1}{2} )</th>
<th>( b \leq -\frac{1}{2} )</th>
<th>( a - c \leq a_1 )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_I )</td>
<td>( a - c \geq a_1 )</td>
<td>( a - c \geq a_1 )</td>
<td>( a - b + \frac{1}{2} )</td>
<td>( a_1 - b + c + \frac{1}{2} )</td>
</tr>
<tr>
<td>( \lambda_{II} )</td>
<td>( a - c \leq a_1 )</td>
<td>( a - c \leq a_1 )</td>
<td>( b - c + 1 )</td>
<td>( a_1 - a + b + 1 )</td>
</tr>
<tr>
<td>( \lambda_{III} )</td>
<td>( b \geq 1 ) or ( b = 0 ) and ( a = -c )</td>
<td>( a - c \leq a_1 )</td>
<td>( a - b + \frac{1}{2} )</td>
<td>( a_1 - b + c + 1 )</td>
</tr>
<tr>
<td>( \lambda_{III} )</td>
<td>( b \leq 0 )</td>
<td>( a - c \leq a_1 )</td>
<td>( b - c + \frac{1}{2} )</td>
<td>( a_1 - a + b + 1 )</td>
</tr>
<tr>
<td>( \lambda_{III} )</td>
<td>( b \leq -\frac{1}{2} )</td>
<td>( a - c \leq a_1 )</td>
<td>( a - b + 1 )</td>
<td>( a_1 - b + c + 1 )</td>
</tr>
</tbody>
</table>

If \( m \) takes negative values in the above table, then we set \( m = 0 \).

Proof: First we observe a symmetry. By sending \( (a, b, c) \mapsto (-c, -b, -a) \) and \( a_i \mapsto a_{5-i+1} \), we send \( \lambda_I \) to \( \lambda_{III} \), and \( (\lambda_{II}, b \geq 1) \) to \( (\lambda_{III}, b \leq -1) \). The multiplicity \( m \) remains unchanged by these transformations.

We will now prove the lemma for \( \lambda_I \) which implies the lemma for \( \lambda_{III} \) by the symmetry.

Again, in order to apply the LR rule we twist representations with \( \det^a/2 \).
Thus, let \( a' = a + n/2, a' = a + (n-1)/2, b' = b + n/2, c' = c + n/2 \) and \( x' = x + n/2 = (n-1)/2 \). First we place the Young diagram \( Y \) of \( V_{\hat{g}_6}(a'+1, a', b', b', c', c') \) into that of \( V_{\hat{g}_6}(a'_1, a'_2, a'_3, \ldots, a'_5) \). Next we will fill the remaining spaces with \( (n-1)/2 \) copies of 1-boxes and 2-boxes. The Young diagram is almost identical to the one in the proof of Lemma 20 except that \( Y \) has one more box in the first row. A check on the diagram shows that \( a = a_2, b = a_3, c = a_4, a_5 = -a_1 \). It remains to fill the shaded region \( x \) and \( y \) with \( (n-1)/2 - c' = -1/2 - c \) 2-boxes.

Suppose \( b \geq 1/2 \). Since \( a + b + c = 0, \pm 1 \), we have

\[
\text{Length of } y \geq -1/2 - c \geq \text{Length of } x
\]

The multiplicity is nonzero if and only if \( a'_1 - b' - 1 \geq -1/2 - c \), that is, \( b - c \leq a_1 - 1/2 \). If it is nonzero, then it equals

\[
\min \left( a - b - \frac{1}{2}, a'_1 - a - \frac{n+1}{2}, a'_1 - \frac{n+1}{2} - b + c \right) + 1.
\]

Finally one checks that the second term is greater than or equal to the third term.
The proof is similar for \( b \leq -1/2 \). This proves the lemma for \( \lambda_I \).

We will not prove the lemma for \( \lambda_{II} \) but we will give an outline. First we consider \( b \leq 0 \). Next we apply the symmetry to get \( b \geq -1 \). Note that the symmetry fails to produce the formula for \( b = 0 \).

**Proof of Lemma 2(ii) for \( E_8 \):** Fix \( a_1 \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z} \). Using (13) we set

\[
\mu(a, b, c) := m(a, b, c; a_1, a_2 = \lambda_2, a_3 = \lambda_4, a_4 = \lambda_6, a_5 = -a_1).
\]

We note that \( a_1 \geq a \geq b \geq c \geq -a_1 \). For any other values \( a, b, c \) where the inequalities does not hold, we set \( \mu(a, b, c) = 0 \). Let

\[
S = \{(\alpha, \beta, \gamma) \in \left(\frac{1}{2} \mathbb{Z} \setminus \mathbb{Z}\right)^3 : a_1 \geq \alpha \geq \beta \geq \gamma \geq -a_1, \alpha + \beta + \gamma = \frac{1}{2}\}
\]

For \((\alpha, \beta, \gamma) \in S\), we define three differences

\[
d_1 = \mu(\alpha + \frac{1}{2}, \beta, \gamma) - \mu(\alpha - \frac{1}{2}, \beta, \gamma),
\]

\[
d_2 = \mu(\alpha, \beta + \frac{1}{2}, \gamma) - \mu(\alpha, \beta - \frac{1}{2}, \gamma),
\]

\[
d_3 = \mu(\alpha, \beta, \gamma - \frac{1}{2}) - \mu(\alpha, \beta, \gamma + \frac{1}{2}).
\]

**LEMMA 22:** \( d_1 + d_2 + d_3 = 0 \).

**Proof:** The lemma is an immediate consequence of the following table.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \alpha - \gamma )</th>
<th>( a_1 )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( d_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta \geq \frac{1}{2} )</td>
<td>( a_1 - \frac{1}{2} )</td>
<td>( a_1 \leq \beta - \gamma - \frac{3}{2} )</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \alpha - \gamma \geq a_1 + \frac{1}{2} )</td>
<td>( a_1 \leq \beta - \gamma - \frac{3}{2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( a_1 \geq \beta - \gamma - \frac{1}{2} )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta \leq -\frac{1}{2} )</td>
<td>( a_1 - \frac{1}{2} )</td>
<td>( a_1 \leq \alpha - \beta - \frac{3}{2} )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \alpha - \gamma \geq a_1 + \frac{1}{2} )</td>
<td>( a_1 \leq \alpha - \beta - \frac{3}{2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( a_1 \geq \alpha - \beta - \frac{1}{2} )</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The proof of the table follows from the tedious case by case checking by hand. We will leave this to the reader.

Lemma 22 implies that the 2-eigenspace and the 0-eigenspace of torus of \( \mathfrak{sl}_{2,s} \) has the same dimension, that is, the trivial representation of \( \mathfrak{sl}_{2,s} \) does not occur. This proves Lemma 2(ii).
References


