

# K-TYPES OF MINIMAL REPRESENTATIONS (*p*-ADIC CASE)

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ABSTRACT. Let  $F$  be a  $p$ -adic field. Let  $G$  be a split simple simply connected group over  $F$  of type  $D_n$ , ( $n \geq 4$ ), or  $E_n$ , ( $n = 6, 7, 8$ ). Let  $K$  be a hyperspecial maximal compact subgroup of  $G$ . In this article we describe  $K$ -types of the minimal representation of  $G$ .

## 1. MINIMAL REPRESENTATION

It will be convenient for us to think of  $G$  as the group of  $F$ -points of a Chevalley group  $\mathbf{G}$ . Let  $\mathcal{R}$  be the ring of integers of  $F$ . Then  $K$  is simply the group of  $\mathcal{R}$ -points of  $\mathbf{G}$ . Let  $\varpi$  be a uniformizing element of  $\mathcal{R}$  and  $\mathcal{R}/\varpi\mathcal{R} \cong \mathbb{F}_q$  the residue field of  $F$ .

Let  $K_1$  be the first principal congruence subgroup. Then  $K/K_1 \cong \mathbb{G}$  is the finite group of  $\mathbb{F}_q$ -points of  $\mathbf{G}$ . Let  $I, K_1 \subset I \subset K$  be an Iwahori subgroup of  $G$ . Then  $I/K_1 \cong \mathbb{B}$ , a Borel subgroup of  $\mathbb{G}$ . Let  $H$  be the Hecke algebra of  $I$ -biinvariant compactly supported functions on  $G$ . The space of  $I$ -fixed vectors of a smooth representation of  $G$  is naturally an  $H$ -module. It is a well known result of Borel [1] that this correspondence defines an equivalence between the category of representations of  $G$  generated by its  $I$ -fixed vectors and the category of representations of  $H$ .

The algebra  $H$  can be described as follows. Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be a set of simple roots. Let  $-\alpha_0$  be the maximal root and let  $\bar{\Delta} = \Delta \cup \{\alpha_0\}$ . Let  $\langle, \rangle$  be a Killing form normalized so that  $\langle \alpha_i, \alpha_i \rangle = 2$  for all  $i$ . Then  $H$  is generated by  $T_i, i = 0, \dots, n$  satisfying the following relations:

$$\begin{aligned} T_i T_j &= T_j T_i & \text{if } \langle \alpha_i, \alpha_j \rangle &= 0; \\ T_i T_j T_i &= T_j T_i T_j & \text{if } \langle \alpha_i, \alpha_j \rangle &= -1; \\ & \text{and } (T_i - q)(T_i + 1) &= 0. \end{aligned}$$

Define an irreducible  $H$ -module  $E$  by (see [7])

$$E = \bigoplus_{i=0}^n \mathbb{C}e_i$$

with the action of  $H$  given by

$$T_i e_j = \begin{cases} -e_j & \text{if } \alpha_i = \alpha_j; \\ qe_j + q^{\frac{1}{2}}e_i & \text{if } \langle \alpha_i, \alpha_j \rangle = -1; \\ qe_j & \text{if } \langle \alpha_i, \alpha_j \rangle = 0. \end{cases}$$

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**Definition 1.1.** *The minimal representation is the unique irreducible representation  $V_{\min}$  such that the space of  $I$ -fixed vectors in  $V_{\min}$  is isomorphic to  $E$ , as an  $H$ -module.*

Repeating previous constructions with  $\Delta$  (instead of  $\bar{\Delta}$ ) gives  $\mathbb{H}$ , the Hecke algebra of  $\mathbb{B}$ -biinvariant functions on  $\mathbb{G}$  and an  $\mathbb{H}$ -module  $\mathbb{E}$  which corresponds to  $V_{\min}$ , the representation of  $\mathbb{G}$  with the minimal dimension (see [3]). It is interesting to note that by setting  $q = 1$ ,  $\mathbb{H}$  becomes the group algebra of the Weyl group of  $G$  and  $\mathbb{E}$  its reflection representation.

Let  $h_1, \dots, h_n$  and  $x_\alpha$ ,  $\alpha \in \Phi$  ( $\Phi$  is the root system) be a Chevalley basis of  $\mathfrak{g}$  as in [6]. Let  $\mathfrak{g}_{\mathbb{Z}}$  be  $\mathbb{Z}$ -span of the Chevalley basis of  $\mathfrak{g}$ . Let

$$\mathfrak{g}_i = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \varpi^i \mathcal{R}.$$

Let  $v_F$  be an evaluation on  $F$  normalized so that  $v_F(\varpi) = 1$ . If  $v_F(p) < p - 1$  then the exponential map is well defined on  $\mathfrak{g}_1$ , and it preserves Haar measures. The groups  $K_i = \exp(\mathfrak{g}_i)$ ,  $i = 1, 2, \dots$  are the principal congruence subgroups. If  $v_F(p) < 1/3(p - 1)$  then the multiplication in  $K_1$  can be defined using the Campbell-Hausdorff formula [9], LG 5.19.

The Killing form  $\langle, \rangle$ , normalized by  $\langle h_i, h_i \rangle = 2$  for all  $i$ , is unimodular on  $\mathfrak{g}_0$  if  $p$  is prime to the determinant of the Cartan matrix (which is 4, 3, 2 and 1 for  $D_n$ ,  $E_6$ ,  $E_7$  and  $E_8$  respectively). Note that  $v_F(p) < 1/3(p - 1)$  implies that  $p \neq 2, 3$ , hence we can assume that the Killing form is unimodular on  $\mathfrak{g}_0$ .

Let  $f_i$  be the characteristic function of  $\mathfrak{g}_i$ . Let  $\psi$  be a non-trivial additive character of  $F$  with conductor  $\mathcal{R}$ . Let  $f$  be a locally constant compactly supported function on  $\mathfrak{g}$ . Define the Fourier transform  $\hat{f}$  of  $f$  by

$$\hat{f}(x) = \int_{\mathfrak{g}} f(y) \psi(\langle x, y \rangle) dy$$

where the Haar measure  $dy$  is normalized so that

$$\hat{f}_i = |\varpi|^{i \dim \mathfrak{g}} f_{-i}.$$

Let  $(\pi, V)$  be an irreducible representation of  $G$ . It defines a distribution  $\Theta_\pi$  as follows. Let  $f$  be a locally constant function supported in  $\mathfrak{g}_1$ . Then

$$\Theta_\pi(f) = \text{tr} \int_{\mathfrak{g}} \pi(\exp x) f(x) dx.$$

A result of Howe and Harish-Chandra [4] says that there is a positive integer  $n_V$  and numbers  $c_{\mathcal{O}}$  such that

$$\Theta_\pi(f) = \sum_{\mathcal{O}} c_{\mathcal{O}} \int \hat{f} \mu_{\mathcal{O}}$$

for every locally constant function  $f$  supported in  $\mathfrak{g}_{n_V}$ . Here the sum is taken over nilpotent orbits and  $\mu_{\mathcal{O}}$  is a  $G$ -invariant measure on  $\mathcal{O}$  constructed as follows. Let  $x \in \mathcal{O}$  and let

$$B_x(y, z) = \langle x, [y, z] \rangle$$

be a bilinear form on  $\mathfrak{g}$ . It induces a non-degenerate symplectic form on  $T_{\mathcal{O}, x}$ , the tangent space of  $\mathcal{O}$  at  $x$ . Then  $\mu_{\mathcal{O}}(x) = |\wedge^d B_x|$  (we shall see on the example of the minimal orbit how this works).

**Remark 1.2.** *One expects that  $n_V = 1$  for representations generated by its  $I$ -fixed vectors. Indeed, Waldspurger has shown this to be true for classical groups [11].*

Let  $D = \max_{c_{\mathcal{O}} \neq 0} \frac{1}{2} \dim \mathcal{O}$ . It follows from the character expansion (see [8]) that the dimension of the space of  $K_i$ -fixed vectors in  $V$  grows as  $q^{D_i}$ .

The minimal non-trivial nilpotent orbit  $\mathcal{O}_{\min}$  is the orbit of  $x_{-\alpha_0}$ . Let  $\Theta_{\min}$  be the character of  $V_{\min}$ . Theorem 2.1 in [8] says that

$$\Theta_{\min}(f) = \int \hat{f} \mu_{\mathcal{O}_{\min}} + c \int \hat{f} \mu_0.$$

Hence the growth of  $\dim V_{\min}^{K_i}$  is the slowest possible, justifying the name “minimal”.

## 2. $K$ -TYPES

The main result is the following.

**Proposition 2.1.** *Assume that  $v_F(p) < 1/3(p-1)$  and  $n_{V_{\min}} = 1$ . Then*

$$V_{\min}|_K = \bigoplus_{i=0}^{\infty} V_i$$

where  $V_i$  are irreducible representations of  $K$  such that  $V_{\min}^{K_{i-1}} \oplus V_i = V_{\min}^{K_i}$ . Here  $K_0 = K$ ,  $K_i$ ,  $i \geq 1$ , are principal congruence subgroups and  $V_{\min}^{K_i}$  is the space of  $K_i$ -fixed vectors. Furthermore  $V_0 = \mathbb{C}$  (the trivial representation of  $K$ ) and  $V_1 = \mathbb{V}_{\min}$ , the minimal representation of  $\mathbb{G}$ , pulled back to  $K$ . We also describe  $V_i$  for  $i > 1$  explicitly.

*Proof.* We first describe  $K$ -types contained in  $V_{\min}^{K_1}$ . Let  $B$  be a Borel subgroup of  $G$ . Since  $G = BK$  and  $V_{\min} \subset \text{ind}_B^G \chi$  for some unramified character  $\chi$ , it follows that

$$V_{\min}^{K_1} \subset C(\mathbb{B} \backslash \mathbb{G})$$

and

$$V_{\min}^I \subset C(\mathbb{B} \backslash \mathbb{G} / \mathbb{B}).$$

Hence the  $K$ -types contained in  $V_{\min}^{K_1}$  are obtained by restricting the  $H$ -module  $E$  to  $\mathbb{H}$ . Since  $E|_{\mathbb{H}} = \mathbb{C} \oplus \mathbb{E}$  the claim follows.

To continue we need to describe  $\mathcal{O}_{\min}$ . Write  $\alpha_0 = m_1 \alpha_1 + \dots + m_n \alpha_n$ . Let  $h_0 = m_1 h_1 + \dots + m_n h_n$ . Then  $(x_{\alpha_0}, h_0, x_{-\alpha_0})$  is an  $sl(2)$ -triple. Let

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h_0, x] = ix\}.$$

Then  $\mathfrak{g} = \bigoplus_{-2 \leq i \leq 2} \mathfrak{g}(i)$ . Assume that  $\alpha_1$  is the unique simple root such that  $\langle \alpha_0, \alpha_1 \rangle \neq 0$ . Then  $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$  is the maximal parabolic subalgebra corresponding to  $\alpha_1$  (see the diagrams on the end of this section). The unipotent radical of  $\mathfrak{p}$  is  $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ . It is a Heisenberg Lie algebra with center  $\mathfrak{g}(2)$ , spanned by  $x_{-\alpha_0}$ . Let

$$\mathfrak{s} = [\mathfrak{g}(0), \mathfrak{g}(0)] \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

It is the centralizer of  $x_{-\alpha_0}$ . Let  $S$  be the centralizer of  $x_{-\alpha_0}$  in  $G$  and let  $\mathbb{S} \subset \mathbb{G}$  the corresponding finite group.

**Lemma 2.2.** *Let  $g = \#\mathbb{G}/\mathbb{S}$  and  $d = 1/2 \dim \mathcal{O}_{\min}$  ( $d=2n-3, 11, 17$  and  $29$ , respectively). If  $i > 2$  then*

$$\dim V_{\min}^{K_i} - \dim V_{\min}^{K_{i-1}} = gq^{d(i-2)}.$$

*Proof.* Let  $\chi_i(x) = |\varpi|^{-i \dim \mathfrak{g}} f_0(\varpi^{-i}x)$ . Then  $P_i = \pi(\chi_i)$  is a projection on  $V_{\min}^{K_i}$ . Since  $\dim V_{\min}^{K_i} = \text{tr}(P_i)$  and  $\hat{\chi}_i(x) = f_{-i}(x)$ , it follows that

$$\dim V_{\min}^{K_i} - \dim V_{\min}^{K_{i-1}} = \int (f_{-i} - f_{-i+1}) \mu_{\mathcal{O}_{\min}}.$$

Write  $\mathcal{O}_{\min} = x_{-\alpha_0}^G$ . Since  $G = KB$ , it follows that  $\mathcal{O}_{\min} = \cup \varpi^i x_{-\alpha_0}^K$ . Hence

$$\mathcal{O}_{\min} \cap (\mathfrak{g}_{-i} \setminus \mathfrak{g}_{-i+1}) = \varpi^{-i} x_{-\alpha_0}^K.$$

Let  $\mathcal{O}_i = \varpi^{-i} x_{-\alpha_0}^K$ . Then  $\mathcal{O}_i$  breaks into  $g$   $K_1$ -orbits

$$\mathcal{O}_i = \cup_{j=1}^g \mathcal{O}_{i,j}.$$

Assume that  $\mathcal{O}_{i,1}$  is the  $K_1$ -orbit of  $x_{-\alpha_0}$ . Hence

$$\dim V_{\min}^{K_i} - \dim V_{\min}^{K_{i-1}} = g \int_{\mathcal{O}_{i,1}} \mu_{\mathcal{O}_{\min}}.$$

To compute the volume of  $\mathcal{O}_{i,1}$  we need to describe precisely the normalization of  $\mu_{\mathcal{O}_{\min}}$  at  $\varpi^{-i}x_{-\alpha_0}$ . Let  $\mathfrak{s}'$  be the span of  $h_0$ ,  $\mathfrak{g}(-2)$  and  $\mathfrak{g}(-1)$ . Then  $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}'$  and the tangent space of  $\mathcal{O}_{\min}$  at  $x_{-\alpha_0}$  can naturally be identified with  $\mathfrak{s}'$ . Note that  $\mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$  is a Heisenberg Lie algebra with center  $\mathfrak{g}(-2)$ , spanned by  $x_{\alpha_0}$ . Let  $\{e_1, \dots, e_{d-1}, f_1, \dots, f_{d-1}\}$  be the part of the Chevalley basis contained in  $\mathfrak{g}(-1)$  labeled so that  $[e_j, f_k] = \delta_{j,k} x_{\alpha_0}$  ( $\delta_{j,k} = 1$  if  $j = k$  and 0 otherwise). We complete it to a basis of  $\mathfrak{s}'$  by adding  $e_d = x_{\alpha_0}$  and  $f_d = \frac{1}{2}h_0$ .

Let  $\mathfrak{s}'_{\mathbb{Z}[1/2]}$  be  $\mathbb{Z}[1/2]$ -span of  $\{e_1, \dots, e_d, f_1, \dots, f_d\}$ . Let  $\mathfrak{g}_{\mathbb{Z}[1/2]}$  and  $\mathfrak{s}_{\mathbb{Z}[1/2]}$  be  $\mathbb{Z}[1/2]$ -span of the Chevalley basis and of its part contained in  $\mathfrak{s}$  respectively. Since  $m_1 = 2$ , and 2 is invertible in  $\mathbb{Z}[1/2]$ ,

$$\mathfrak{g}_{\mathbb{Z}[1/2]} = \mathfrak{s}_{\mathbb{Z}[1/2]} \oplus \mathfrak{s}'_{\mathbb{Z}[1/2]}.$$

Let

$$\mathfrak{s}'_i = \mathfrak{s}_{\mathbb{Z}[1/2]} \otimes_{\mathbb{Z}[1/2]} \varpi^i \mathcal{R}.$$

and let  $S'_i = \exp \mathfrak{s}'_i$ ,  $i = 1, 2, \dots$ . Define analogously  $\mathfrak{s}_i$  and  $S_i$ . Also, since  $p$  is odd

$$\mathfrak{g}_i = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \varpi^i \mathcal{R} = \mathfrak{g}_{\mathbb{Z}[1/2]} \otimes_{\mathbb{Z}[1/2]} \varpi^i \mathcal{R}.$$

Hence  $\mathfrak{g}_i = \mathfrak{s}_i \oplus \mathfrak{s}'_i$ ,  $K_i = S_i S'_i$ ,  $i = 1, 2, \dots$  and we have a sequence of measure-preserving bijections:

$$\mathcal{O}_{1,j} \cong S'_1 \cong \mathfrak{s}'_1.$$

Since  $\mathfrak{s}'_1$  is the  $\mathcal{R}$ -span of  $\varpi e_j$  and  $\varpi f_j$   $j = 1, \dots, d$ , and

$$|B_{\varpi^{-i}x_{-\alpha_0}}(\varpi e_j, \varpi f_k)| = \delta_{j,k} q^{(i-2)}$$

it follows that the volume of  $\mathcal{O}_{i,1}$  is  $q^{d(i-2)}$ . The lemma is proved.

Recall [5], Lemma 1.2, that irreducible representations of  $K_1/K_i$  are parametrized by  $K_1$ -orbits in  $\mathfrak{g}_{-i}/\mathfrak{g}_{-1}$ . To describe the correspondence, recall that the Fourier transform defines an isomorphism between the spaces of functions  $C(\mathfrak{g}_1/\mathfrak{g}_i)$  and  $C(\mathfrak{g}_{-i}/\mathfrak{g}_{-1})$ . Let  $\mathcal{O}$  be a  $K_1$ -orbit in  $\mathfrak{g}_{-i}/\mathfrak{g}_{-1}$  and  $E_{\mathcal{O}}$  the corresponding irreducible representation of  $K_1$ . Then

- (1) the character of  $E_{\mathcal{O}}$ , pulled back to  $\mathfrak{g}_1$  via  $\log$ , is the Fourier transform of the characteristic function of  $\mathcal{O}$  divided by  $\#\mathcal{O}^{1/2}$ .
- (2)  $\dim E_{\mathcal{O}} = \#\mathcal{O}^{1/2}$ ; this is a consequence of (1).

Let  $\mathcal{O}_{i,j} \subset \mathcal{O}_i$ , ( $i \geq 2$ ) be a  $K_1$ -orbit. Then  $\mathcal{O}'_{i,j} = \mathcal{O}_{i,j} + \mathfrak{g}_{-1}$  is a  $K_1$ -orbit in  $\mathfrak{g}_{-i}/\mathfrak{g}_{-1}$ . Since  $\mathcal{O}'_{i,j}$  are  $K$ -conjugated and  $\#\mathcal{O}_{i,1} = \#K_1/S_1K_{i-1} = q^{2d(i-2)}$ , it follows that

- (1)  $\dim E_{\mathcal{O}'_{i,j}} = q^{d(i-2)}$ .
- (2)  $E_{\mathcal{O}_{i,j}}$  is a summand of  $V_{\min}$ .

From this and Lemma 2.2 we conclude that

$$V_{\min}^{K_i} = V_{\min}^{K_{i-1}} \oplus (\oplus_{j=1}^g E_{\mathcal{O}'_{i,j}}).$$

Let  $S_0 = S \cap K$ . Obviously,  $S_0$  preserves  $E_{\mathcal{O}'_{i,1}} \subset V_{\min}^{K_i}$ . Let

$$V_i = \text{ind}_{S_0K_1}^K E_{\mathcal{O}'_{i,1}}.$$

Since

$$V_i|_{K_1} = \oplus_{j=1}^g E_{\mathcal{O}'_{i,j}}$$

it follows from the Mackey's irreducibility criterion ([10] Prop. 23) that  $V_i$  is irreducible. Also, by the Frobenius reciprocity  $V_i \subset V_{\min}$ . The proposition is proved.

We proceed to write down  $V_i$ , the irreducible representations of  $K/K_i$ . Let  $\psi_i$  be a character of  $\mathfrak{g}$  defined by  $\psi_i(x) = \psi(\langle x, \varpi^{-i}x_{-\alpha_0} \rangle)$ . If  $j \leq i \leq 2j$  then  $K_j/K_i \cong \mathfrak{g}_j/\mathfrak{g}_i$  (this follows from the Campbell-Hausdorff formula). Hence  $\psi_i$  defines a character of  $K_j/K_i$ . We first describe  $E_{\mathcal{O}'_{i,1}}$ . We have two cases.

- (1)  $i$  is even. Write  $i = 2j$ . Then  $\psi_i$  defines a character of  $K_j/K_i$ . Note that  $\psi_i|_{S_j} = 1$ . Therefore, since  $S_1$  centralizes  $\psi_i$ , we can extend  $\psi_i$  to  $S_1K_j$  by  $\psi_i|_{S_1} = 1$ . Then

$$E_{\mathcal{O}'_{i,1}} = \text{ind}_{S_1K_j}^{K_1} \psi_i.$$

- (2)  $i$  is odd. Write  $i = 2j + 1$ . Then  $\psi_i$  defines a character of  $K_{j+1}/K_i$ . As in the even case extend  $\psi_i$  to  $S_1K_{j+1}$ . Then  $S_1K_j/\ker \psi_i$  is a Heisenberg group. Let  $\rho_i$  be the corresponding irreducible representation such that the center acts via the character  $\psi_i$ . Then

$$E_{\mathcal{O}'_{i,1}} = \text{ind}_{S_1K_j}^{K_1} \rho_i.$$

We know a priori that  $E_{\mathcal{O}'_{i,1}}$  extends to  $S_0$ . Since  $[\mathbb{S}, \mathbb{S}] = \mathbb{S}$  the extension is unique in view of the following lemma (take  $A = S_0K_1$  and  $B = K_1$ ).

**Lemma 2.3.** *Let  $A$  be a group and  $B$  a normal subgroup. Let  $C = A/B$ . Assume that  $[C, C] = C$ . Let  $\rho$  be an irreducible finite-dimensional representation of  $B$ . If  $\rho$  extends to  $A$  then it extends uniquely.*

*Proof.* Let  $\rho_1$  and  $\rho_2$  be two extensions. By the Schur Lemma, for any  $a \in A$  there exists a scalar  $\chi(a)$  such that  $\rho_1(a) = \chi(a)\rho_2(a)$ . Obviously,  $\chi$  is a character of  $C$ . Since  $[C, C] = C$ , it must be trivial. The lemma is proved.

We now give precise definitions of  $V_i$ ,  $i = 2, 3, \dots$ . Again we have two cases.

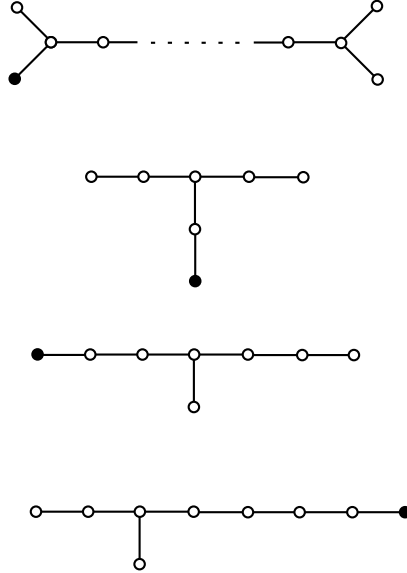
- (1)  $i$  is even. Write  $i = 2j$ . Then  $\psi_i$  defines a character of  $K_j/K_i$ . Extend  $\psi_i$  to  $S_0K_j$  by  $\psi_i|_{S_0} = 1$ . Then

$$V_i = \text{ind}_{S_0K_j}^K \psi_i.$$

- (2)  $i$  is odd. Write  $i = 2j + 1$ . Then  $\psi_i$  defines a character of  $K_{j+1}/K_i$ . Extend  $\psi_i$  to  $S_1K_{j+1}$  by  $\psi_i|_{S_1} = 1$ . Then  $S_1K_j/\ker \psi_i$  is a Heisenberg group. Let  $\rho_i$  be the corresponding irreducible representation such that the center acts via the character  $\psi_i$ . It extends to  $S_0$  via the usual Weil representation argument. Then

$$V_i = \text{ind}_{S_0K_j}^K \rho_i.$$

We conclude this paper by giving some explicit data. Recall that the extended Dynkin diagram for  $G$  is a graph obtained from the set of roots  $\bar{\Delta}$  by connecting  $\alpha_i$  and  $\alpha_j$  if and only if  $\langle \alpha_i, \alpha_j \rangle = -1$ :



The black vertex in each diagram corresponds to the root  $\alpha_0$ .

Recall that  $\alpha_1$  is the unique simple root such that  $\langle \alpha_0, \alpha_1 \rangle = -1$ . Since  $\mathbb{S}$  is a semi-direct product of a Heisenberg group of order  $q^{2d-1}$ , and a semi-simple, simply connected group with the Dynkin diagram obtained by removing  $\alpha_1$  from the Dynkin diagram of  $\mathbb{G}$ , one can compute  $g = \#\mathbb{G}/\mathbb{S}$  using formulas in [2], page 75. The answers are:

$$\begin{aligned} D_n & \frac{(q^n - 1)(q^{2n-4} - 1)(q^{2n-2} - 1)}{(q^2 - 1)(q^{n-2} - 1)} \\ E_6 & \frac{(q^8 - 1)(q^9 - 1)(q^{12} - 1)}{(q^3 - 1)(q^4 - 1)} \\ E_7 & \frac{(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)}{(q^4 - 1)(q^6 - 1)} \\ E_8 & \frac{(q^{20} - 1)(q^{24} - 1)(q^{30} - 1)}{(q^6 - 1)(q^{10} - 1)} \end{aligned}$$

Finally, we note that

$$\dim \mathbb{V}_{\min} = gq/(q^{d+1} - 1)$$

$d = 1/2 \dim \mathcal{O}_{\min}$ , and it is  $2n - 3$ , 11, 17 and 29 respectively.

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