MATH 5210, HW III
SOLUTIONS

1) Let (Y, d) be a complete metric space and X a dense subset of Y. The set X is a also a metric space with respect to the same metric. Let X^* be the completion of X. Recall that X^* is the set of equivalence classes of Cauchy sequences (x_n) in X. Since Y is complete, $\lim_n x_n$ exists in Y. Equivalent Cauchy sequences have the same limit, hence $f((x_n)) = \lim_n x_n$ is a well defined map $f : X^* \to Y$. Show that f is an isomorphism of metric spaces.

Solution. Here is an example before the proof: $Y = [0, 1]$ and $X = (0, 1]$. Then $(x_n) = (1/n)$ is Cauchy sequence in $(0, 1]$ with the limit 0, thus $f((1/n)) = 0$. In particular this exercise proves that the completion of $(0, 1]$ is $[0, 1]$.

We show that f is surjective. Let $x \in Y$. Since X is dense in Y there exists a sequence (x_n) in X converging to x. Thus $f((x_n)) = x$ and f is surjective. To show that f is an isomorphism of X^* and Y it remains to prove that f preserves the distance. (Observe that any distance preserving function is one to one.) Recall that the distance d^* between two Cauchy sequences (x_n) and (y_n) is

$$d^*((x_n), (y_n)) = \lim_n d(x_n, y_n).$$

Let $x = f((x_n)) = \lim_n x_n \in Y$ and $y = f(y_n)) = \lim_n y_n \in Y$. Since distance is a continuous function,

$$d(x, y) = \lim_n d(x_n, y_n).$$

This shows that $d^*((x_n), (y_n)) = d(x, y)$ as desired.

2) Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Prove that the set of piece-wise linear function (i.e. whose graphs are obtained by connecting the dots in the plane) is dense in V, with respect to the sup norm, that is, for every $f \in V$ and every $\epsilon > 0$, there exists a piece-wise linear function g such that $|f(x) - g(x)| < \epsilon$ for all $x \in [0, 1]$. Hint: use uniform continuity of f.

Solution. Since f is uniformly continuous, for every $\epsilon > 0$ there exist a natural number n such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < 1/n$. Let $g(x)$ be a piece-wise linear function whose graph is the union of n segments connecting the points

$$(0, f(0)),\ (1/n, f(1/n)),\ \ldots (1, f(1)).$$

On each segment $[(i - 1)/n, i/n]$, $i = 1, \ldots, n$, it is easy to see that $|f(x) - g(x)| < \epsilon$ hence the same holds on $[0, 1]$.

3) Fix $K(x, y)$, a continuous function on $[0, 1]^2$. Let $f(x)$ be a continuous function on $[0, 1]$. Let

$$g(x) = \int_0^1 K(x, y)f(y)\ dy.$$

Prove that $g(x)$ is a continuous function on $[0,1]$. Hint: K is uniformly continuous, why? Let $V = C([0,1])$ be the space of continuous functions on $[0,1]$. Consider V as a normed space with the sup norm. Let $T : V \to V$, $T(f) = g$ for every $f \in V$, as above. Prove that T is bounded.

Solution. Let $||f||$ denote the sup norm of f. To prove that g is continuous, consider the difference

$$g(x) - g(z) = \int_0^1 (K(x,y) - K(z,y))f(y)\, dy.$$

So the goal is to show that, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x) - g(z)| < \epsilon$ if $|x - z| < \delta$. Since K is uniformly continuous, there exists $\delta > 0$ such that

$$|K(x,y) - K(z,y)| < \epsilon/||f||,$$

for all y, if $|x - z| < \delta$. Using this bound in the above integral and $|f(y)| \leq ||f||$ one shows that $|g(x) - g(z)| < \epsilon$. Thus T is a well defined linear operator $T : V \to V$. To prove that T is bounded we shall use that K is a bounded function, since it is a continuous function on a compact set. Thus there exists $C \geq 0$ such $|K(x,y)| \leq C$ for all $(x,y) \in [0,1]^2$. Then

$$|g(x)| \leq \int_0^1 C \cdot ||f||\, dy = C||f||.$$

Hence $||g|| = \sup_{x\in[0,1]} |g(x)| \leq C||f||$, so T is bounded.

4) Let U be a dense subspace of a normed space V. Let $g : U \to \mathbb{R}$ be a bounded linear functional i.e. there exists $C \geq 0$ such that

$$|g(x)| \leq C||x||$$

for all $x \in U$. Then g can be extended (uniquely) to a linear functional $f : V \to \mathbb{R}$ satisfying the same bound. Hint: any $x \in V$ is a limit of a Cauchy sequence (x_n) in U.

Solution. Let $x \in V$. Let (x_n) be a Cauchy sequence in U such that $x = \lim_n x_n$. The idea is to define $f(x) = \lim_n g(x_n)$, to that end, we need to prove that $(g(x_n))$ is a Cauchy sequence of real numbers. This is not automatic from continuity of g. For example, the continuous function $h : (0,1] \to \mathbb{R}$, $g(x) = 1/x$ sends the Cauchy sequence $(1/n)$ to the sequenece (n) of natural numbers, which is not Cauchy. And h does not extend to a continuous function on $[0,1]$, the completion of $(0,1]$. However, since g is linear and bounded,

$$|g(x_n) - g(x_m)| = |g(x_n - x_m)| < C||x_n - x_m||$$

so $(g(x_n))$ is Cauchy since (x_n) is. Thus we have extended g to f on the whole of V. We need to show that f is linear. If $x = \lim x_n$ and $y = \lim_n y_n$, then $x + y = \lim_n (x_n + y_n)$. Hence

$$f(x) + f(y) = \lim_n g(x_n) + \lim_n g(y_n) = \lim_n g(x_n + y_n) = f(x + y)$$

from the usual properties of limits. A similar argument proves that $f(\lambda x) = \lambda f(x)$ for any $\lambda \in \mathbb{R}$ and $x \in V$. It remains to prove the bound:

$$|f(x)| = |\lim_n g(x_n)| = \lim_n |g(x_n)| \leq \lim_n C||x_n|| = C||x||.$$

5) Recall the normed space $\ell^2(\mathbb{N})$, the set of all infinite tuples of real numbers $x = (x_1, x_2, \ldots)$ such that $||x||^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$, with the norm $||x||$ so defined. Let $S \subset \ell^2(\mathbb{N})$ be the subset of all x with $x_i \in \mathbb{Q}$ and almost all $x_i = 0$. This is a countable set. Prove that S is dense.

Solution. Let $\epsilon > 0$. There exists n such that $\sum_{i>n} x_i^2 < \epsilon^2$. Then $||x - z|| < \epsilon$ where

$$z = (x_1, \ldots, x_n, 0, 0, \ldots).$$

Let $y = (y_1, \ldots, y_n, 0, 0, \ldots)$ with y_i rational. Pick y_i close to x_i such that $||z - y|| < \epsilon$. Hence $||x - y|| < 2\epsilon$ by the triangular inequality.

6) Let V be a normed space, and $A, B \subset V$ two open sets. Prove that

$$A + B = \{x + y \mid x \in A, y \in B\}$$

is open.

Solution. The easiest way to prove this is as follows. Observe that, for every $v \in V$, the translation map $t_v : V \to V$, $t_v(x) = x + v$ for all $x \in V$, is an isometry of V. Thus a translate of a ball is a ball, of an open set is an open set etc. In particular, for every $b \in B$, $A + b$ is an open set. Since union of open sets is open,

$$A + B = \cup_{b \in B} A + b$$

is open. Observe that we never used that B is open in this argument. Thus the conclusion is valid if only one of the two sets is open.

7) Perhaps you have seen the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Where does this come from? The purpose of this exercise is to derive this formula as a special case of the Parseval's identity. Let $X = (-1/2, 1/2]$. Let $f(x) = x$ on X. Compute $||f||^2$, the square of $L^2(X)$ norm of f. Then Fourier expand f and then compute $||f||^2$ using the Parseval's identity. (Be careful, the norm of $\sin(2\pi nx)$ is not 1). Deduce the identity.

Solution. Parseval's idenity: If $u_1, u_2, \ldots$ is an orthogonal basis of a Hilbert space V, then for every $v \in V$,

$$||v||^2 = \sum_{n=1}^{\infty} \frac{(v, u_n)^2}{||u_n||^2}.$$

(Note that this must be right, since it is invariant under rescaling of u_n, and for $(u_n, u_n) = 1$ one gets the simple form of the identity.) Mathematics is the art of substitution, and this is what we do here, with $V = L^2(X)$, v is the function $f(x) = x$ and u_n are trigonometric functions. Since x is an odd function we only need $u_n = \sin(2\pi nx)$. We computed in a lecture that $||u_n||^2 = \frac{1}{2}$. Next, using integration by parts,

$$(x, u_n) = \int_{-1/2}^{1/2} x \sin(2\pi nx)\ dx = \frac{1}{2\pi n}$$

On the other hand,

$$||x||^2 = \int_{-1/2}^{1/2} x^2 \, dx = \frac{1}{12}.$$

Substitute to finish.

8) Let $M \geq 0$. Let c_n be a sequence of real numbers such that $|c_n| \leq M/n^2$ for all n. Then the series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(2\pi n t)$$

converges uniformly, for all $t \in \mathbb{R}$. Hence f is a periodic and continuous function f. Prove that the series converges to f in $L^2((-1/2, 1/2])$ that is

$$\lim_n ||f - f_n|| = 0$$

where f_n is the sequence of partial sums, and $||\cdot||$ the L^2-norm. Hint: use Lebesgue dominated convergence theorem.

Solution. In fact this is much easier than what I suggested. The series $\sum_{n=1}^{\infty} \frac{M}{n^2}$ is convergent, hence given $\epsilon > 0$ there exists N such that $\sum_{n>N}^{\infty} \frac{M}{n^2} < \epsilon$. Since $|\sin(2\pi n t)| \leq 1$, it follows that

$$|f(t) - f_N(t)| < \epsilon$$

for all t. Hence

$$||f - f_N||^2 = \int_{-1/2}^{1/2} |f - f_N|^2 < \epsilon^2.$$

9) Let V be a Hilbert space. Let $W \subset V$ be a closed subspace. Prove that W contains a dense countable set, so it is also a Hilbert space. Hint: consider the projection $P : V \to W$.

Solution. Since V is a Hilbert space it contains a dense countable set S. We claim that $P(S) \subset W$ is a dense set in W. Let $w \in W$. Since S is dense, there exists a sequence (v_n) in S converging to w. Since P is continuous,

$$P(w) = P(\lim_n v_n) = \lim_n P(v_n).$$

But $P(w) = w$, hence w is the limit of the sequence $(P(v_n))$. Hence $P(S)$ is dense in W.