1) Please justify all steps in the following:

(1) Prove that \( f(x) = x \exp\left(-\frac{x^2}{2}\right) \) is in \( L^1(\mathbb{R}) \).

(2) Use Theorem 2.27 to compute the derivative of \( g(t) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \cos(tx) \, dx \), where \( t \in \mathbb{R} \).

(3) Use \( \exp\left(-\frac{x^2}{2}\right)' = -x \exp\left(-\frac{x^2}{2}\right) \) and integration by parts to show that \( g(t) \) satisfies a first order differential equation. Solve it to find \( g(t) \).

Solution: (1) The function \( f \) is continuous hence measurable. For integrality, note that \( f \) is odd and positive for \( x \geq 0 \), so it suffices to show integrality on \([0, \infty)\). For every natural number \( n \), let \( f_n = f \cdot \chi_{[0,n]} \), where \( \chi_{[0,n]} \) is the characteristic function of \([0,n]\). This is a monotone sequence of positive functions converging pointwise to \( f \) for \( x \geq 0 \). Next, we use that the Lebesgue integral of \( f \) on \([0,n]\) is equal to the Riemann integral, to compute

\[
\int_{-\infty}^{\infty} f_n(x) \, dx = \int_{0}^{n} f(x) \, dx = -\exp\left(-\frac{x^2}{2}\right)|_0^n = 1 - \exp\left(-\frac{n^2}{2}\right).
\]

It follows, from the monotone convergence theorem, that \( f(x) \) is integrable for \( x \geq 0 \) with integral equal to 1. (2) Fix \( t \) and an integer \( n \) such that \( t \) is contained in the interior of \([-n,n]\). Let \( h(x,t) = \exp\left(-\frac{x^2}{2}\right) \cos(tx) \). Since

\[
\frac{\partial}{\partial t} h(x,t) = -x \exp\left(-\frac{x^2}{2}\right) \sin(tx)
\]

is bounded by \( |f| \) for all \( t \in [-n,n] \),

\[
g'(t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} h(x,t) \, dx = \int_{-\infty}^{\infty} -x \exp\left(-\frac{x^2}{2}\right) \sin(tx) \, dx.
\]

(3) Using integration by parts for Riemann integral on the segment \([0,n]\),

\[
\int_{0}^{n} -x \exp\left(-\frac{x^2}{2}\right) \sin(tx) \, dx = \exp\left(-\frac{x^2}{2}\right) \sin(tx)|_0^n - \int_{0}^{n} t \exp\left(-\frac{x^2}{2}\right) \cos(tx) \, dx.
\]

Using the dominated convergence theorem on \([0,\infty)\), we can take the limit \( n \to \infty \), which yields \( g'(t) = -tg(t) \). Thus \( g(t) = C \exp\left(-\frac{t^2}{2}\right) \) and

\[
C = g(0) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \, dx = \sqrt{2\pi}.
\]

2) Let \( \mathcal{M} \subseteq \mathcal{P}(X) \) be the \( \sigma \) algebra generated by an elementary family \( \mathcal{E} \), see the definition on page 23. Let \( \mu \) and \( \nu \) be two positive finite measures on \( \mathcal{M} \), such that \( \mu(E) = \nu(E) \) for all \( E \) in \( \mathcal{E} \). Prove that \( \mu = \nu \).

Solution: Let \( \mathcal{A} \) be the algebra consisting of finite unions of pairwise disjoint elements in \( \mathcal{E} \). It is clear that \( \mu(E) = \nu(E) \) for all \( E \) in \( \mathcal{A} \). Now there are two ways to continue. One: Let \( \mathcal{N} \)
be the set of all \( E \in X \) such that \( \mu(E) = \nu(E) \). Then \( \mathcal{N} \) is a monotone class and it contains \( \mathcal{A} \). Hence \( \mathcal{N} = \mathcal{M} \). The other: \( \mu \) and \( \nu \) are pre-measures on \( \mathcal{A} \). Any pre-measure on \( \mathcal{A} \) uniquely extends to a measure on \( \mathcal{M} \). Hence \( \mu = \nu \) on \( \mathcal{M} \).

In the following exercises \( \mu \) denotes the standard Lebesgue measure on the Borel algebra of \( \mathbb{R}^n \). Let \( P^n \) be the open punctured ball \( 0 < ||x|| < 1 \) and \( S^{n-1} \) the sphere \( ||x|| = 1 \), where \( ||x|| \) is the standard norm on \( \mathbb{R}^n \). Note that \( P^n \) is homeomorphic to \((0,1) \times S^{n-1} \). Let \( d\nu \) denote the standard Lebesgue measure on \((0,1)\).

3) Take \( n=2 \) above. Let \( d\varphi \) be the Lebesgue measure on the unit circle \( S^1 \) (the arc length). Let \( d\varphi \) be the product measure on the Borel algebra of \((0,1) \times S^1 \approx P^2 \). Prove that for every Borel set \( E \) in \( P^2 \),

\[
\mu(E) = \int_E r \, dr \, d\varphi.
\]

Solution: Let \( \nu(E) = \int_E r \, dr \, d\varphi \). Then \( \nu \) and \( \mu \) are two measure on the Borel algebra of \( P^2 \). By problem 2), it suffices to show that \( \mu(E) = \nu(E) \) on elementary sets which, in this case, can be taken to be products

\[
(a,b] \times [\alpha, \beta] = (0,b] \times [\alpha, \beta] \setminus (0,a] \times [\alpha, \beta]
\]
i.e. differences of pizza slices of radii \( b \) and \( a \), and the same angle \( \beta - \alpha \), where other shapes of brackets are also allowed. By Fubini,

\[
\nu((0,b] \times [\alpha, \beta]) = \int_a^b \int_\alpha^\beta \, rd\varphi \, dr = \frac{1}{2} b^2 (\beta - \alpha)
\]

and that is indeed the measure of the pizza slice. (If we take these measures as granted. A proper way how to decompose the Lebesgue measure on \( P^n \) in polar coordinates is the following two exercises.)

4) Let \( F \) be a Borel set in \( S^{n-1} \). Then \( c(F) = (0,1) \times F \) is a Borel set in \( P^n \). (Why?) Let \( \nu(F) = n\mu(c(F)) \). Prove that \( \nu \) is a Borel measure on \( S^{n-1} \) invariant under the orthogonal transformations.

Solution: Since \( P^n \) is homeomorphic to \((0,1) \times S^{n-1} \) the Borel algebra of \( P^n \) is generated by products of Borel sets in \((0,1)\) and in \( S^{n-1} \), hence \( c(F) \) is a Borel set in \( P^n \). The invariance of the Lebesgue measure for an orthogonal transformation \( g \) (Corollary 2.46) implies that \( \mu(g(c(F))) = \mu(c(F)) \). Since \( g(c(F)) = c(g(F)) \), it follows that \( \nu(F) = \nu(g(F)) \). That \( \nu \) is a measure is easy, it follows from the observation that if \( F \) is a disjoint union of \( F_i, i = 1, 2, \ldots \) then \( c(F) \) is a disjoint union of \( c(F_i), i = 1, 2, \ldots \). Hence \( \sigma \)-additivity of \( \nu \) is a consequence of \( \sigma \)-additivity of \( \mu \).

5) Let \( \nu \) be the Borel measure on \( S^{n-1} \) as in 4). Let \( dr \otimes \nu \) denote the product measure on the Borel algebra of \((0,1) \times S^{n-1} \approx P^n \). Prove that for every Borel set \( E \) in \( P^n \),

\[
\mu(E) = \int_E r^{n-1} dr \otimes \nu.
\]
Hint: It suffices to do this for $E = (0, a) \times F$, where $F$ is a Borel set in $S^{n-1}$ (why?) and in this case it reduces to proving that $\mu((0, a) \times F) = a^n\mu((0, 1) \times F)$. Why is the last identity true?

Solution: Let $\nu(E) = \int_E r^{n-1}dr \otimes \nu$. Then $\nu$ and $\mu$ are two measures on the Borel algebra of $P^n$. In order to show that $\mu = \nu$, it suffices to do it on an elementary family generating the Borel algebra. And we take the elementary family to consist of $(a, b] \times F$, where $F$ is any Borel set in $S^{n-1}$. Furthermore, as in problem 3), we can reduce to checking $\nu(E) = \mu(E)$ to $E = (0, b] \times F$. By Fubini

$$\nu(E) = \int_0^b \int_F r^{n-1}dr \otimes \nu = \frac{b^n}{n} \nu(F).$$

On the other hand, $(0, b] \times F$ is the image of $(0, 1] \times F$ under the linear map $v \mapsto b \cdot v$. By Theorem 2.44

$$\mu((0, b] \times F) = b^n\mu((0, 1] \times F)$$

and the right hand side is equal to $\frac{b^n}{n} \nu(F)$ by the definition of $\nu$.

The problem 5) combined with 14) on page 52 (the previous HW) gives us the formula for integration in “polar” coordinates in $\mathbb{R}^n$, first on any ball of a finite radius $n$ and then by monotone convergence in general.