SOLUTIONS TO HW III FOR MATH 6210

1) Please justify all steps in the following:

(1) Prove that $f(x) = x\exp(-\frac{x^2}{2})$ is in $L^1(\mathbb{R})$.

(2) Use Theorem 2.27 to compute the derivative of $g(t) = \int_{\mathbb{R}} \exp(-\frac{x^2}{2})\cos(tx)dx$, where $t \in \mathbb{R}$.

(3) Use $\exp(-\frac{x^2}{2})' = -x\exp(-\frac{x^2}{2})$ and integration by parts to show that $g(t)$ satisfies a first order differential equation. Solve it to find g.

Solution: (1) The function f is continuous hence measurable. For integrality, note that f is odd and positive for $x \geq 0$, so it suffices to show integrality on $[0, \infty)$. For every natural number n, let $f_n = f \cdot \chi_{[0,n]}$, where $\chi_{[0,n]}$ is the characteristic function of $[0, n]$. This is a monotone sequence of positive functions converging pointwise to f for $x \geq 0$. Next, we use that the Lebesgue integral of f on $[0, n]$ is equal to the Riemann integral, to compute

$$\int_{\mathbb{R}} f_n(x)dx = \int_0^n f(x)dx = -\exp(-\frac{x^2}{2})|_0^n = 1 - \exp(-\frac{n^2}{2}).$$

It follows, from the monotone convergence theorem, that $f(x)$ is integrable for $x \geq 0$ with integral equal to 1. (2) Fix t and an integer n such that t is contained in the interior of $[-n, n]$. Let $h(x, t) = \exp(-\frac{x^2}{2})\cos(tx)$. Since

$$\frac{\partial}{\partial t}h(x, t) = -x\exp(-\frac{x^2}{2})\sin(tx)$$

is bounded by $|f|$ for all $t \in [-n, n]$,

$$g'(t) = \int_{\mathbb{R}} \frac{\partial}{\partial t}h(x, t)dx = \int_{\mathbb{R}} -x\exp(-\frac{x^2}{2})\sin(tx)dx.$$

(3) Using integration by parts for Riemann integral on the segment $[0, n]$,

$$\int_0^n -x\exp(-\frac{x^2}{2})\sin(tx)dx = \exp(-\frac{x^2}{2})\sin(tx)|_0^n - \int_0^n t\exp(-\frac{x^2}{2})\cos(tx)dx.$$

Using the dominated convergence theorem on $[0, \infty)$, we can take the limit $n \to \infty$, which yields $g'(t) = -tg(t)$. Thus $g(t) = C\exp(-\frac{t^2}{2})$ and

$$C = g(0) = \int_{\mathbb{R}} \exp(-\frac{x^2}{2})dx = \sqrt{2\pi}.$$

2) Let $\mathcal{M} \subseteq \mathcal{P}(X)$ be the σ algebra generated by an elementary family $\mathcal{E}$, see the definition on page 23. Let μ and ν be two positive finite measures on $\mathcal{M}$, such that $\mu(E) = \nu(E)$ for all E in $\mathcal{E}$. Prove that $\mu = \nu$.

Solution: Let $\mathcal{A}$ be the algebra consisting of finite unions of pairwise disjoint elements in $\mathcal{E}$. It is clear that $\mu(E) = \nu(E)$ for all E in $\mathcal{A}$. Now there are two ways to continue. One: Let $\mathcal{N}$

be the set of all $E \in X$ such that $\mu(E) = \nu(E)$. Then $\mathcal{N}$ is a monotone class and it contains $\mathcal{A}$. Hence $\mathcal{N} = \mathcal{M}$. The other: μ and ν are pre-measures on $\mathcal{A}$. Any pre-measure on $\mathcal{A}$ uniquely extends to a measure on $\mathcal{M}$. Hence $\mu = \nu$ on $\mathcal{M}$.

In the following exercises μ denotes the standard Lebesgue measure on the Borel algebra of $\mathbb{R}^n$. Let P^n be the open punctured ball $0 < ||x|| < 1$ and S^{n-1} the sphere $||x|| = 1$, where $||x||$ is the standard norm on $\mathbb{R}^n$. Note that P^n is homeomorphic to $(0,1) \times S^{n-1}$. Let dr denote the standard Lebesgue measure on $(0,1)$.

3) Take $n = 2$ above. Let $d\varphi$ be the Lebesgue measure on the unit circle S^1 (the arc length). Let $dr\ d\varphi$ be the product measure on the Borel algebra of $(0,1) \times S^1 \cong P^2$. Prove that for every Borel set E in P^2,

$$\mu(E) = \int_E r dr\ d\varphi.$$

Solution: Let $\nu(E) = \int_E r dr\ d\varphi$. Then ν and μ are two measure on the Borel algebra of P^2. By problem 2), it suffices to show that $\mu(E) = \nu(E)$ on elementary sets which, in this case, can be taken to be products

$$(a,b] \times [\alpha,\beta] = (0,b] \times [\alpha,\beta] \setminus (0,a] \times [\alpha,\beta]$$

i.e. differences of pizza slices of radii b and a, and the same angle $\beta - \alpha$, where other shapes of brackets are also allowed. By Fubini,

$$\nu((0,b] \times [\alpha,\beta]) = \int_0^b \int_\alpha^\beta r d\varphi\ dr = \frac{1}{2} b^2 (\beta - \alpha)$$

and that is indeed the measure of the pizza slice. (If we take these measures as granted. A proper way how to decompose the Lebesgue measure on P^n in polar coordinates is the following two exercises.)

4) Let F be a Borel set in S^{n-1}. Then $c(F) = (0,1) \times F$ is a Borel set in P^n. (Why?) Let $\nu(F) = n\mu(c(F))$. Prove that ν is a Borel measure on S^{n-1} invariant under the orthogonal transformations.

Solution: Since P^n is homeomorphic to $(0,1) \times S^{n-1}$ the Borel algebra of P^n is generated by products of Borel sets in $(0,1)$ and in S^{n-1}, hence $c(F)$ is a Borel set in P^n. The invariance of the Lebesuge measure for an orthogonal transformation g (Corollary 2.46) implies that $\mu(g(c(F)) = \mu(c(F))$. Since $g(c(F)) = c(g(F))$, it follows that $\nu(F) = \nu(g(F))$. That ν is a measure is easy, it follows from the observation that if F is a disjoint union of F_i, $i = 1, 2, \ldots$ then $c(F)$ is a disjoint union of $c(F_i)$, $i = 1, 2, \ldots$. Hence σ-additivity of ν is a consequence of σ-additivity of μ.

5) Let ν be the Borel measure on S^{n-1} as in 4). Let $dr \otimes \nu$ denote the product measure on the Borel algebra of $(0,1) \times S^{n-1} \cong P^n$. Prove that for every Borel set E in P^n,

$$\mu(E) = \int_E r^{n-1} dr \otimes \nu.$$

Hint: It suffices to do this for $E = (0, a) \times F$, where F is a Borel set in S^{n-1} (why?) and in this case it reduces to proving that $\mu((0, a) \times F) = a^n \mu((0, 1) \times F)$. Why is the last identity true?

Solution: Let $\nu(E) = \int_E r^{n-1} dr \otimes \nu$. Then ν and μ are two measures on the Borel algebra of P^n. In order to show that $\mu = \nu$, it suffices to do it on an elementary family generating the Borel algebra. And we take the elementary family to consist of $(a, b] \times F$, where F is any Borel set in S^{n-1}. Furthermore, as in problem 3), we can reduce to checking $\nu(E) = \mu(E)$ to $E = (0, b] \times F$. By Fubini

$$\nu(E) = \int_0^b \int_F r^{n-1} dr \otimes \nu = \frac{b^n}{n} \nu(F).$$

On the other hand, $(0, b] \times F$ is the image of $(0, 1] \times F$ under the linear map $v \mapsto b \cdot v$. By Theorem 2.44

$$\mu((0, b] \times F) = b^n \mu((0, 1] \times F)$$

and the right hand side is equal to $\frac{b^n}{n} \nu(F)$ by the definition of ν.

The problem 5) combined with 14) on page 52 (the previous HW) gives us the formula for integration in "polar" coordinates in $\mathbb{R}^n$, first on any ball of a finite radius n and then by monotone convergence in general.