HW III FOR MATH 6210
SOLUTIONS

1) Let V be a normed space and U a closed subspace. For every $x \in V$, let $||x+U|| = \inf\{||x+y|| \mid y \in U\}$.

(1) Prove that $||x+U||$ is a norm on V/U.
(2) If V is complete, so is V/U.
(3) Let $P : V \to V/U$ be $P(x) = x+U$. Prove that P has norm 1.

Solution: (1) If $||x+U|| = 0$, then there is a sequence $x_n = x + y_n \in x+U$ such whose limit is 0. But this mens then limit of y_n is $-x$, hence $x \in U$ since U is closed. Hence $x+U = U$ i.e. we have a norm, not a semi-norm. The second property is trivial, triangle inequality: Let $\epsilon > 0$. There exists $x' \in x+U$ and $y' \in y+U$ such that $||x'|| < ||x+U|| + \epsilon/2$ and $||y'|| < ||y+U|| + \epsilon/2$. Then

$$||x+y+U|| \leq ||x'+y'|| \leq ||x'|| + ||y'|| \leq ||x+U|| + ||y+U|| + \epsilon.$$

(2) Let $\sum_i x_i + U$ be absolutely converging series in V/U. For every i let $x_i' \in x_i + U$ such that $||x_i'|| < ||x+U|| + 1/2^i$. Then $\sum_i x_i'$ is an absolutely converging series in V, so it has a limit, say x. Then $x+U$ is the limit of $\sum_i x_i + U$ by continuity of the projection map P, which is obviously bounded i.e. $||P|| \leq 1$. (3) The easiest way to prove this is to observe that $||x+U|| < 1$ iff there exists $x' \in x+U$ such that $||x'|| < 1$. In other words P maps the open unit ball in V onto the open unit ball in V/U.

2) Let $T : V \to W$ be a bounded map. Let U be the kernel of T.

(1) Prove that U is closed.
(2) Prove that $||S|| = ||T||$ where $S : V/U \to W$ such that $S(x+U) = T(x)$, for all $x \in V$.

Solution: (1) Since T is continuous, $U = T^{-1}(0)$ is a closed set. (2) Observe that it is convenient to define $||T||$ as the supremum of $||T(x)||$ for $||x|| < 1$. Now it is clear that $||T|| = ||S||$, see (3) of the first exercise.

3) Let $T : V \to W$ be a map between two normed spaces, where W is finite dimensional. Let U be the kernel of T. Prove that T is bounded if U is closed.

Solution: If U is closed then, by the first exercise, V/U is a normed space under the norm $||x+U|| = \inf\{||x+y|| \mid y \in U\}$. The induced map $S : V/U \to W$ is automatically continuous, since it is a map between two finite-dimensional spaces. Now T is bounded since it is a composite of two bounded maps, P and S, where P is as in the first exercise, or $||T|| = ||S||$ using the second exercise.

4) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E \in \mathcal{M}$ of finite and positive measure. Let $T : L^1(X) \to \mathbb{R}$ be defined by $T(f) = \int_E f$. Prove that T is a bounded functional, and compute its norm.

Solution: $|T(f)| = |\int_E f| \leq \int_E |f| \leq \int_X |f| = ||f||$. Hence $|T(f)| \leq ||f||$, but this equality is achieved for $f = \chi_E$, hence $||T|| = 1$.

5) Let $T : V \to U$ be a bounded map between two normed spaces. Let $T^* : U^* \to V^*$ be defined by $T^*(f) = f \circ T$ for all $f \in U^*$ (the adjoint map).

(1) Prove that $||T^*|| = ||T||$.
(2) If we identify V and U with their canonical images in V^{**} and U^{**} prove that the restriction of T^{**} to V coincides with T.

Solution: (1) $||T^*(f)|| = ||f \circ T|| \leq ||f|| \cdot ||T||$ (this is true for a composite of any two bounded operators) hence $||T^*|| \leq ||T||$. Let $\epsilon > 1$. Then there exists $v \in V$, $||v|| = 1$ such that $||T(v)|| > ||T|| - \epsilon$. By Hahn-Banach, there exists $f \in U^*$ such that $||f|| = 1$ and $T^*(f)(v) = f(T(v)) = ||T(v)||$. Thus $||T^*(f)|| \geq ||T(v)||$ and $||T^*|| \geq ||T(v)||$ since $||T^*||$ is the supremum of $||T^*(f)||$ over all $||f|| = 1$. Hence $||T^*|| > ||T|| - \epsilon$, for any $\epsilon > 0$, and this is the other inequality.

6) Let $V = C([0, 1])$ be the space of continuous functions equipped with the sup norm. Let $U = C^1([0, 1])$ be the subspace of continuously differentiable functions. Let $T : U \to V$ be defined by $T(f) = f'$. Prove that the graph of T is closed, and that T is not bounded. Let V be a space complete with respect to two norms $|| \cdot ||_1 \leq || \cdot ||_2$. Prove that the norms are equivalent.

Solution: Let $||f||$ denote the sup norm of $f \in V$. The space V is a Banach space with respect to this norm, while U is a (dense) subspace. The operator T is unbounded, since $||x^n|| = 1$ but $||T(x^n)|| = ||nx^{n-1}|| = n$, for $n = 0, 1, 2, \ldots$. The space U is a also a Banach space, however, with respect to $||f|| + ||f'||$. On $U \times V$ we can take the norm $||(f, g)|| = ||f|| + ||g||$. Hence on the graph (f, f') this gives $||f|| + ||f'||$, but U is complete under this norm, so the graph is closed.

7) Let $T : V \to U$ be a linear map between two Banach spaces such that if f is a continuous functional on U, then $f \circ T$ is a continuous functional on V. Prove that T is bounded. Hint: prove that the graph of T is closed, to that end use Theorem 5.8 c on page 159.

Solution: Let $(v_n, u_n) \in V \times U$ be a Cauchy sequence in the graph of T, i.e. $T(v_n) = u_n$ for all n. Since $V \times U$ is complete (equivalently both V and U are complete) there exists $(v, u) \in V \times U$, the limit of the sequence (v_n, u_n). In other words, v is the limit of v_n and u of u_n. We need to prove that $T(v) = u$. If not then $T(v) - u$ is a non-zero element of U, and by Hahn-Banach there exists a continuous functional f on U such that $f(T(u) - v) \neq 0$. On the other hand, $f(T(u_n) - v_n) = 0$ for all n and $f \circ T$ is continuous, hence $f(T(u) - v) = 0$, a contradiction.