SOLUTIONS TO HW II FOR MATH 6210

14, page 52. To prove that λ is a measure, we need to check two properties. The first is $\lambda(\emptyset) = 0$. This follows from $f \cdot \chi_\emptyset = 0$. Next, let $E = \cup_{n=1}^\infty E_n$ be a disjoint union of measurable sets. The second property is $\lambda(E) = \sum_{n=1}^\infty \lambda(E_n)$. To that end, let $g = f \cdot \chi_E$ and $g_n = f \cdot \chi_{E_n}$. Then the positive series $\sum_{n=1}^\infty g_n$ converges monotonely to g. Now the second property follows by a simple application of the monotone convergence theorem.

To check that $\int g \, d\lambda = \int gf \, d\mu$, after it has been checked for simple g, let ϕ_n be a monotone sequence of positive simple functions converging to g. Then $\phi_n f$ is a monotone sequence of measurable functions converging to gf. Hence, by applying the monotone convergence theorem twice,

$$\int g \, d\lambda = \lim_{n\to\infty} \int \phi_n \, d\lambda = \lim_{n\to\infty} \int \phi_n f \, d\mu = \int gf \, d\mu.$$

15, page 52. Apply the monotone convergence theorem to the sequence $g_n = f_1 - f_n$. Remark: Problem 6) in the first HW is a special case, take f_n to be the characteristic function of A_n.

16, page 52. First solution. $\int f$ is the supremum of $\int \phi$ where ϕ is a simple function such that $f \geq \phi$. Let ϕ be such that $\int \phi > \int f - \epsilon$. Let E be the support of ϕ i.e. the set of points x such that $\phi(x) > 0$. From the definition of simple functions E has a finite measure, and

$$\int_E f \geq \int_E \phi = \int \phi > \int f - \epsilon.$$

Second solution. Let $E_n = \{x | f(x) > 1/n\}$. Let $f_n = f \cdot \chi_{E_n}$. Then f_n is a monotone sequence converging to f point-wise. By the monotone convergence theorem, there exists n such that

$$\int f_n = \int_{E_n} f > \int f - \epsilon.$$

It remains to show that $\mu(E_n)$ is finite. Since $f > \frac{1}{n}\chi_{E_n}$,

$$\int f > \int \frac{1}{n}\chi_{E_n} = \frac{1}{n}\mu(E_n),$$

the claim follows.

23, page 59, part b. We need to show that $h = g$ and $H = G$ a.e. where g and G are defined as supremum and infimum, respectively, of sequences of simple functions g_n and G_n computing the Riemann lower and upper sums arising from a nested sequence of divisions $t_i < t_{i+1}$ of $(a, b]$, one for each n, where $\sup(t_{i+1} - t_i)$ goes to 0, as n increases. Let E be the set of all end points of the intervals (t_i, t_{i+1}) for all n. This is a countable set, hence of Lebesgue measure 0, and we shall show that $h(x) = g(x)$ for x not in E. So fix x not in E. For every $\delta > 0$, let $h_\delta(x) = \inf_{|x-y|<\delta} f(y)$. It suffices to show that for every $\delta > 0$ there exists n such that $g_n(x) \geq h_\delta(x)$ and, conversely, for every n there exists $\delta > 0$ such that $h_\delta(x) \geq g_n(x)$. The first inequality: given $\delta > 0$, take n large enough such that $t_{i+1} - t_i < \delta$ for all segments in the division defining g_n. The second inequality: given n, let (t_i, t_{i+1}) be the segment in the corresponding division containing x. Take $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (t_i, t_{i+1})$. (And this can be done if x is not in E.)

Since f is Riemann integrable iff $g = G$ a.e. the upshot of this problem is that f is Riemann integrable iff $h = H$ a.e. hence iff the set of points where f is not continuous is of the Lebesgue measure 0.

26, page 60. First solution. Let x_n be a sequence converging to x. Then (I am assuming here that $x_n < x$, a similar argument works for $x_n > x$)

$$F(x) - F(x_n) = \int_{(x_n,x]} f(x)\ dx = \int f_n(x)\ dx$$

where $f_n = f \cdot \chi_{(x_n,x]}$. The sequence f_n is dominated by f and the limit is the delta function at x, whose integral is 0, since $\mu(\{x\}) = 0$. Hence $\lim_{n\to\infty} F(x_n) = F(x)$ be the dominated convergence theorem. Second solution. If we write $f = f^+ - f^-$ where f^+ and f^- are positive functions, then $F = F^+ - F^-$, hence it suffices to prove continuity of F assuming that f is positive. But then f defines a finite measure on $\mathbb{R}$, as in the problem 14 above. Hence F is continuous by problem 7) in the first HW, since $\mu(\{x\}) = 0$ for the standard measure on $\mathbb{R}$.