14, page 52. To prove that \( \lambda \) is a measure, we need to check two properties. The first is \( \lambda(\emptyset) = 0 \). This follows from \( f \cdot \chi_\emptyset = 0 \). Next, let \( E = \bigcup_{n=1}^{\infty} E_n \) be a disjoint union of measurable sets. The second property is \( \lambda(E) = \sum_{n=1}^{\infty} \lambda(E_n) \). To that end, let \( g = f \cdot \chi_E \) and \( g_n = f \cdot \chi_{E_n} \). Then the positive series \( \sum_{n=1}^{\infty} g_n \) converges monotonely to \( g \). Now the second property follows by a simple application of the monotone convergence theorem.

To check that \( \int g \, d\lambda = \int g \, d\mu \), after it has been checked for simple \( g \), let \( \phi_n \) be a monotone sequence of positive simple functions converging to \( g \). Then the positive series \( \sum_{n=1}^{\infty} \phi_n \) converges monotonely to \( \phi \). Hence, by applying the monotone convergence theorem twice,

\[
\int g \, d\lambda = \lim_{n \to \infty} \int \phi_n \, d\lambda = \lim_{n \to \infty} \int \phi_n \, d\mu = \int g \, d\mu.
\]

15, page 52. Apply the monotone convergence theorem to the sequence \( g_n = f_1 - f_n \). Remark: Problem 6) in the first HW is a special case, take \( f_n \) to be the characteristic function of \( A_n \).

16, page 52. First solution. \( \int f \) is the supremum of \( \int \phi \) where \( \phi \) is a simple function such that \( f \geq \phi \). Let \( \phi \) be such that \( \int \phi > \int f - \epsilon \). Let \( E \) be the support of \( \phi \), i.e. the set of points \( x \) such that \( \phi(x) > 0 \). From the definition of simple functions \( E \) has a finite measure, and

\[
\int_E f \geq \int_E \phi = \int \phi > \int f - \epsilon.
\]

Second solution. Let \( E_n = \{ x | f(x) > 1/n \} \). Let \( f_n = f \cdot \chi_{E_n} \). Then \( f_n \) is a monotone sequence converging to \( f \) point-wise. By the monotone convergence theorem, there exists \( n \) such that

\[
\int f_n = \int_{E_n} f > \int f - \epsilon.
\]

It remains to show that \( \mu(E_n) \) is finite. Since \( f > \frac{1}{n} \chi_{E_n} \),

\[
\int f > \int \frac{1}{n} \chi_{E_n} = \frac{1}{n} \mu(E_n),
\]

the claim follows.

23, page 59, part b. We need to show that \( h = g \) and \( H = G \) a.e. where \( g \) and \( G \) are defined as supremum and infimum, respectively, of sequences of simple functions \( g_n \) and \( G_n \) computing the Riemann lower and upper sums arising from a nested sequence of divisions \( t_i < t_{i+1} \) of \( (a, b] \), one for each \( n \), where \( \sup(t_{i+1} - t_i) \) goes to 0, as \( n \) increases. Let \( E \) be the set of all end points of the intervals \( (t_i, t_{i+1}) \) for all \( n \). This is a countable set, hence of Lebesgue measure 0, and we shall show that \( h(x) = g(x) \) for \( x \) not in \( E \). So fix \( x \) not in \( E \). For every \( \delta > 0 \), let \( h_\delta(x) = \inf_{|x-y|<\delta} g(y) \). It suffices to show that for every \( \delta > 0 \) there exists \( n \) such that \( g_n(x) \geq h_\delta(x) \) and, conversely, for every \( n \) there exists \( \delta > 0 \) such that \( h_\delta(x) \geq g_n(x) \). The
first inequality: given $\delta > 0$, take $n$ large enough such that $t_{i+1} - t_i < \delta$ for all segments in the division defining $g_n$. The second inequality: given $n$, let $(t_i, t_{i+1})$ be the segment in the corresponding division containing $x$. Take $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (t_i, t_{i+1})$. (And this can be done if $x$ is not in $E$.)

Since $f$ is Riemann integrable iff $g = G$ a.e. the upshot of this problem is that $f$ is Riemann integrable iff $h = H$ a.e. hence iff the set of points where $f$ is not continuous is of the Lebesgue measure 0.

26, page 60. First solution. Let $x_n$ be a sequence converging to $x$. Then (I am assuming here that $x_n < x$, a similar argument works for $x_n > x$)

$$F(x) - F(x_n) = \int_{(x_n, x]} f(x) \, dx = \int f_n(x) \, dx$$

where $f_n = f \cdot \chi_{(x_n, x]}$. The sequence $f_n$ is dominated by $f$ and the limit is the delta function at $x$, whose integral is 0, since $\mu(\{x\}) = 0$. Hence $\lim_{n \to \infty} F(x_n) = F(x)$ be the dominated convergence theorem. Second solution. If we write $f = f^+ - f^-$ where $f^+$ and $f^-$ are positive functions, then $F = F^+ - F^-$, hence it suffices to prove continuity of $F$ assuming that $f$ is positive. But then $f$ defines a finite measure on $\mathbb{R}$, as in the problem 14 above. Hence $F$ is continuous by problem 7) in the first HW, since $\mu(\{x\}) = 0$ for the standard measure on $\mathbb{R}$. 