HW II FOR MATH 6210, SOLUTIONS

In the following exercises X is always a measure space, i.e. there is a sigma algebra $\mathcal{M} \subset P(X)$ and a measure μ on it.

1) Assume that $X = A \cup B$ where $A, B \in \mathcal{M}$. Let $f : X \to \mathbb{R}$. Prove that X is measurable if and only if the restrictions of f to A and B are measurable.

2) Let $f \in L^+(X)$ such that $\int f < \infty$. Prove that, for every $\epsilon > 0$, there exists a set of finite measure $E \subset X$ such that

$$\int_E f > \int f - \epsilon.$$

First solution. $\int f$ is the supremum of $\int \phi$ where ϕ is a simple function such that $f \geq \phi$. Let ϕ be such that $\int \phi > \int f - \epsilon$. Let E be the support of ϕ i.e. the set of points x such that $\phi(x) > 0$. From the definition of simple functions E has a finite measure, and

$$\int_E f \geq \int_E \phi = \int \phi > \int f - \epsilon.$$

Second solution. Let $E_n = \{x | f(x) > 1/n\}$. Let $f_n = f \cdot \chi_{E_n}$. Then f_n is a monotone sequence converging to f point-wise. By the monotone convergence theorem, there exists n such that

$$\int f_n = \int_{E_n} f > \int f - \epsilon.$$

It remains to show that $\mu(E_n)$ is finite. Since $f > \frac{1}{n}\chi_{E_n}$,

$$\int f > \int \frac{1}{n}\chi_{E_n} = \frac{1}{n}\mu(E_n),$$

the claim follows.

3) Suppose a sequence $\{f_n\}$ in $L^+(X)$ decreases pointwise to f, and $\int f_1 < \infty$. Prove that $\lim_{n\to\infty} \int f_n = \int f$. (Do not use the dominated convergence theorem or Fatou's lemma.)

Solution: Apply the monotone convergence theorem to $g_i = f_1 - f_i$.

4) Let $f \in L^+(X)$. Then $\lambda(E) = \int_E f d\mu$ is a measure on $\mathcal{M}$ and for every $g \in L^+(X)$, $\int g d\lambda = \int g f d\mu$. Hint: do this for simple g firstly.

Solution: To prove that λ is a measure, we need to check two properties. The first is $\lambda(\emptyset) = 0$. This follows from $f \cdot \chi_\emptyset = 0$. Next, let $E = \cup_{n=1}^\infty E_n$ be a disjoint union of measurable sets. The second property is $\lambda(E) = \sum_{n=1}^\infty \lambda(E_n)$. To that end, let $g = f \cdot \chi_E$ and $g_n = f \cdot \chi_{E_n}$. Then the positive series $\sum_{n=1}^\infty g_n$ converges monotonely to g. Now the second property follows by a simple application of the monotone convergence theorem.

To check that $\int g \, d\lambda = \int gf \, d\mu$, after it has been checked for simple g, let ϕ_n be a monotone sequence of positive simple functions converging to g. Then $\phi_n f$ is a monotone sequence of measurable functions converging to gf. Hence, by applying the monotone convergence theorem twice,

$$\int g\ d\lambda = \lim_{n\to\infty} \int \phi_n\ d\lambda = \lim_{n\to\infty} \int \phi_n f\ d\mu = \int gf\ d\mu.$$

5) Let $\{f_n\} \in L^+(X)$ converging pointwise to f. Assume that $\int f_n \to \int f < \infty$. Then $\int_E f_n \to \int_E f$ for every $E \in \mathcal{M}$.

Solution: By the Fatou's lemma $\int_E f \leq \text{liminf}_n \int_E f_n$ and $\int_{E^c} f \leq \text{liminf}_n \int_{E^c} f_n$. By properties of limits,

$$\text{limsup}_n \int_E f_n = \text{limsup}_n (\int f_n - \int_{E^c} f_n) = \int f - \text{liminf}_n \int_{E^c} f_n \leq \int f - \int_{E^c} f = \int_E f.$$

For example, the second equality follows from $\text{limsup}_n (a_n + b_n) = \text{lim}_n a_n + \text{limsup}_n b_n$, for a convergent sequence $\{a_n\}$ and a sequence $\{b_n\}$. Thus

$$\text{limsup}_n \int_E f_n \leq \int_E f \leq \text{liminf}_n \int_E f_n,$$

as desired.

6) Prove that $f(x) = x\exp(-\frac{x^2}{2})$ is Lebesgue integrable on $[0, \infty)$ and compute its integral.

Solution: The function f is continuous hence measurable. For every natural number n, let $f_n = f \cdot \chi_{[0,n]}$, where $\chi_{[0,n]}$ is the characteristic function of $[0, n]$. This is a monotone sequence of positive functions converging pointwise to f for $x \geq 0$. Next, we use that the Lebesgue integral of f on $[0, n]$ is equal to the Riemann integral, to compute

$$\int_{\mathbb{R}} f_n(x)dx = \int_0^n f(x)dx = -\exp(-\frac{x^2}{2})|_0^n = 1 - \exp(-\frac{n^2}{2}).$$

It follows, from the monotone convergence theorem, that $f(x)$ is integrable for $x \geq 0$ with integral equal to 1.

7) Please justify all steps in the following:

(1) Compute the derivative of $g(t) = \int_{\mathbb{R}} \exp(-\frac{x^2}{2}) \cos(tx)dx$, where $t \in \mathbb{R}$.
(2) Use $\exp(-\frac{x^2}{2})' = -x\exp(-\frac{x^2}{2})$ and integration by parts to show that $g(t)$ satisfies a first order differential equation. Solve it to find g.

(1) Fix t and an integer n such that t is contained in the interior of $[-n, n]$. Let $h(x, t) = \exp(-\frac{x^2}{2}) \cos(tx)$. Note that

$$\frac{\partial}{\partial t} h(x, t) = -x\exp(-\frac{x^2}{2}) \sin(tx)$$

is bounded by $|f|$ for all $t \in [-n, n]$, where f is from the previous exercise. Since $|f|$ is integrable, the conditions of Theorem 2.27 are satisfied, hence

$$g'(t) = \int_{\mathbb{R}} \frac{\partial}{\partial t} h(x, t)dx = \int_{\mathbb{R}} -x\exp(-\frac{x^2}{2}) \sin(tx)dx.$$

(2) Using integration by parts for the Riemann integral on $[-n, n]$,

$$\int_{-n}^{n} -x \exp(-\frac{x^2}{2}) \sin(tx) dx = \exp(-\frac{x^2}{2}) \sin(tx)|_{-n}^{n} - \int_{-n}^{n} t \exp(-\frac{x^2}{2}) \cos(tx) dx.$$

Using the dominated convergence theorem on $\mathbb{R}$, taking $n \to \infty$ yields $g'(t) = -tg(t)$. Thus $g(t) = C \exp(-\frac{t^2}{2})$ and

$$C = g(0) = \int_{\mathbb{R}} \exp(-\frac{x^2}{2}) dx = \sqrt{2\pi}.$$