HW I FOR MATH 6210
PARTIAL SOLUTIONS

1) Let $\mathfrak{M}$ be an infinite σ-algebra. Prove that

- $\mathfrak{M}$ contains an infinite sequence of disjoint sets.
- $\mathrm{card}(\mathfrak{M}) \geq \mathfrak{c}$.

Two solutions to the first bullet. First: Let X be the ambient set. Let $A \in \mathfrak{M}$ be non-trivial. Since $\mathfrak{M}$ is infinite, at least one of the families $\{A \cap B\}$ or $\{A^c \cap B\}$, as B runs through $\mathcal{M}$, is infinite. Assume the latter i.e. we have an infinite σ-algebra $\mathcal{M}_{A^c}$ on A^c consisting of all intersections $A^c \cap B$. Set $A_1 = A$ and repeat the process with A^c and infinite σ-algebra $\mathcal{M}_{A^c}$ to construct $A_2 \subset A^c$ etc... Second: Let P be a partition of X into disjoint sets in $\mathfrak{M}$. For two such partitions we write $P \leq P'$ if P' is a refinement of P. Note that such finite partition P cannot be maximal, otherwise $\mathfrak{M}$ would be finite. Thus there is an infinite sequence of partitions

$$P_1 \leq P_2 \leq \ldots.$$

For every $x \in X$, let $A_n \in P_n$ be the class of x. Let $C_x = \cap_n A_n$. Then $C_x \in \mathfrak{M}$, since $\mathfrak{M}$ is a σ-algebra, and $\{C_x\}$ is an infinite partition of X.

2) Let $\mathcal{E}_1 \subset P(X_1)$ and $\mathcal{E}_2 \subset P(X_2)$ two elementary families. Let $\mathcal{E}_1 \times \mathcal{E}_2$ be the collection of all products $E_1 \times E_2$ where $e_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$. Prove that this is an elementary family i of sets in $X_1 \times X_2$.

3) An algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countably increasing unions i.e. if $E_1 \subseteq E_2 \subseteq \ldots \in \mathcal{A}$ then $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$. Similarly, a finitely additive measure μ on a σ-algebra $\mathcal{M}$ is a measure if and only if for any $E_1 \subseteq E_2 \subseteq \ldots \in \mathcal{M}$, $\lim_{i \to \infty} \mu(E_i) = \mu(\cup_{i=1}^{\infty} E_i)$.

4) Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N}$ be the collection of all sets $N \in \mathcal{M}$ of measure 0. Let $\overline{\mathcal{M}}$ be the collection of unions $E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. Define $\bar{\mu}(E \cup F) = \mu(E)$. Prove that $(X, \overline{\mathcal{M}}, \bar{\mu})$ is a measure space.

5) Let $X = (0, 1]$. Let $\mathcal{A}$ be the algebra consisting of finite disjoint unions of intervals $(a, b]$, with the usual pre-measure defined by $\mu_0((a, b]) = b - a$. Let μ^* be the corresponding outer measure. Let $A \subset X$. Prove that $\mu^*(A) + \mu^*(A^c) = 1$ if and only if for every $\epsilon > 0$, there exists $E \in \mathcal{A}$ such that $\mu^*(A \Delta E) < \epsilon$.

Solution: Assume $\mu^*(A) + \mu^*(A^c) = 1$. Let $\epsilon > 0$. Let $\cup_i E_i$ and $\cup_i F_i$ be covers of A and A^c such that

$$\sum_i \mu(E_i) \leq \mu^*(A) + \epsilon \text{ and } \sum_i \mu(F_i) \leq \mu^*(A^c) + \epsilon.$$

Write $\cup_i E_i = E \cup E'$ and $\cup_i F_i = F \cup F'$ where E and F are unions of the first N terms, and E' and F' the tails. We pick N large enough so that $\mu^*(E'), \mu^*(F') < \epsilon$. Adding up above inequalities, and using $\mu^*(A) + \mu^*(A^c) = 1$, it follows that $\mu(E) + \mu(F) \leq 1 + 2\epsilon$. The complement of $E \cup F$ is covered by E' and F', hence $\mu(E \cup F) > 1 - 2\epsilon$. Combining these,

$$\mu(E \cap F) = \mu(E) + \mu(F) - \mu(E \cup F) < 4\epsilon.$$

Since $A\Delta E$ is covered by E', F' and $(E \cap F)$ it follows that $\mu^*(A\Delta E) < 6\epsilon$.

In the opposite direction, note that $A\Delta E = A^c\Delta E^c$. Since $A \subseteq E \cup A\Delta E$ and $A^c \subseteq E^c \cup A^c\Delta E^c$, we have

$$\mu^*(A) + \mu^*(A^c) \leq \mu^*(E) + \mu^*(A\Delta E) + \mu^*(E^c) + \mu^*(A^c\Delta E^c) < 1 + 2\epsilon.$$

6) The setting as in the previous exercise. Prove that $A \subset X$ is measurable if and only if $\mu^*(A) + \mu^*(A^c) = 1$.

7) Let μ be a finite Borel measure on $\mathbb{R}$. Let $F(x) = \mu(-\infty, x]$. Show that $F(x)$ is continuous at x if and only if $\mu(\{x\}) = 0$.

Solution: If $x_n \to x$ from the right, then $(-\infty, x] = \cap_n(-\infty, x_n]$ which shows that $F(x)$ is right continuous. If $x_n \to x$ from left, then $(-\infty, x) = \cup_n(-\infty, x_n]$ which shows that $\lim_{n\to\infty} F(x_n) = F(x) - \mu(\{x\})$. Hence left continuity holds iff $\mu(\{x\}) = 0$.