1) Let $\mathcal{M}$ be an infinite $\sigma$-algebra. Prove that
- $\mathcal{M}$ contains an infinite sequence of disjoint sets.
- $\text{card}(\mathcal{M}) \geq c$.

2) Let $\mathcal{E}_1 \subset P(X_1)$ and $\mathcal{E}_2 \subset P(X_2)$ two elementary families. Let $\mathcal{E}_1 \times \mathcal{E}_2$ be the collection of all products $E_1 \times E_2$ where $e_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$. Prove that this is an elementary family i of sets in $X_1 \times X_2$.

3) An algebra $\mathcal{A}$ is a $\sigma$-algebra if and only if it is closed under countably increasing unions i.e. if $E_1 \subseteq E_2 \subseteq \ldots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. Similarly, a finitely additive measure $\mu$ on a $\sigma$-algebra $\mathcal{M}$ is a measure if and only if for any $E_1 \subseteq E_2 \subseteq \ldots \in \mathcal{M}$, $\lim_{i \to \infty} \mu(E_i) = \mu(\bigcup_{i=1}^{\infty} E_i)$.

4) Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N}$ be the collection of all sets $N \in \mathcal{M}$ of measure 0. Let $\overline{\mathcal{M}}$ be the collection of unions $E \cup F$ where $E \in \mathcal{M}$ and $F \subseteq N$ for some $N \in \mathcal{N}$. Define $\overline{\mu}(E \cup F) = \mu(E)$. Prove that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is a measure space.

5) Let $X = (0, 1]$. Let $\mathcal{A}$ be the algebra consisting of finite disjoint unions of intervals $(a, b]$, with the usual pre-measure defined by $\mu_0((a, b]) = b - a$. Let $\mu^*$ be the corresponding outer measure. Let $A \subset X$. Prove that $\mu^*(A) + \mu^*(A^c) = 1$ if and only if for every $\epsilon > 0$, there exists $E \in \mathcal{A}$ such that $\mu^*(A \Delta E) < \epsilon$.

6) The setting as in the previous exercise. Prove that $A \subset X$ is measurable if and only if $\mu^*(A) + \mu^*(A^c) = 1$.

7) Let $\mu$ be a finite Borel measure on $\mathbb{R}$. Let $F(x) = \mu(-\infty, x]$. Show that $F(x)$ is continuous at $x$ if and only if $\mu(\{x\}) = 0$. 
