**MATH 5210, HW III**
**DUE APRIL 20**

1) Let $(Y, d)$ be a complete metric space and $X$ a dense subset of $Y$. The set $X$ is a also a metric space with respect to the same metric. Let $X^*$ be the completion of $X$. Recall that $X^*$ is the set of equivalence classes of Cauchy sequences $(x_n)$ in $X$. Since $Y$ is complete, $\lim_n x_n$ exists in $Y$. Equivalent Cauchy sequences have the same limit, hence $f((x_n)) = \lim_n x_n$ is a well defined map $f : X^* \to Y$. Show that $f$ is an isomorphism of metric spaces.

2) Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Prove that the set of piece-wise linear function (i.e. whose graphs are obtained by connecting the dots in the plane) is dense in $V$, with respect to the sup norm, that is, for every $f \in V$ and every $\epsilon > 0$, there exists a piece-wise linear function $g$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in [0, 1]$. Hint: use uniform continuity of $f$.

3) Fix $K(x, y)$, a continuous function on $[0, 1]^2$. Let $f(x)$ be a continuous function on $[0, 1]$. Let

$$g(x) = \int_0^1 K(x, y)f(y)\ dy.$$

Prove that $g(x)$ is a continuous function on $[0, 1]$. Hint: $K$ is uniformly continuous, why? Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Consider $V$ as a normed space with the sup norm. Let $T : V \to V$, $T(f) = g$ for every $f \in V$, as above. Prove that $T$ is bounded.

4) Let $U$ be a dense subspace of a normed space $V$. Let $g : U \to \mathbb{R}$ be a bounded linear functional i.e. there exists $C \geq 0$ such that

$$|g(x)| \leq C||x||$$

for all $x \in U$. Then $g$ can be extended (uniquely) to a linear functional $f : V \to \mathbb{R}$ satisfying the same bound. Hint: any $x \in V$ is a limit of a Cauchy sequence $(x_n)$ in $U$.

5) Recall the normed space $\ell^2(\mathbb{N})$, the set of all infinite tuples of real numbers $x = (x_1, x_2, \ldots)$ such that $||x||^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$, with the norm $||x||$ so defined. Let $S \subset \ell^2(\mathbb{N})$ be the subset of all $x$ with $x_i \in \mathbb{Q}$ and almost all $x_i = 0$. This is a countable set. Prove that $S$ is dense.

6) Let $V$ be a normed space, and $A, B \subset V$ two open sets. Prove that

$$A + B = \{x + y \mid x \in A, y \in B\}$$

is open.

7) Perhaps you have seen the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Where does this come from? The purpose of this exercise is to derive this formula as a special case of the Parseval's identity. Let $X = (-1/2, 1/2]$. Let $f(x) = x$ on $X$. Compute $||f||^2$, the square of $L^2(X)$ norm of $f$. Then Fourier expand $f$ and then compute $||f||^2$ using the Parseval's identity. (Be careful, the norm of $\sin(2\pi nx)$ is not 1). Deduce the identity.

8) Let $M \geq 0$. Let $c_n$ be a sequence of real numbers such that $|c_n| \leq M/n^2$ for all $n$. Then the series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(2\pi nt)$$

converges uniformly, for all $t \in \mathbb{R}$. Hence $f$ is a periodic and continuous function $f$. Prove that the series converges to $f$ in $L^2((-1/2, 1/2])$ that is

$$\lim_n ||f - f_n|| = 0$$

where $f_n$ is the sequence of partial sums, and $||\cdot||$ the $L^2$-norm. Hint: use Lebesgue dominated convergence theorem.

9) Let $V$ be a Hilbert space. Let $W \subset V$ be a closed subspace. Prove that $W$ contains a dense countable set, so it is also a Hilbert space. Hint: consider the projection $P : V \to W$.