MATH 5210, HW I
DUE FRIDAY, JAN 31

1) In this and the following problem, use $d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$ as the distance function on $\mathbb{R}^2$. Use ϵ-δ definition of continuity to prove that the multiplication map $\mathbb{R}^2 \to \mathbb{R}$ is continuous.

2) Let $p_i : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the i-th coordinate. Prove that p_i is continuous. Let (X, d) be a metric space. Let $f : X \to \mathbb{R}^2$ be a map, and write $f(x) = (f_1(x), f_2(x))$ for every $x \in X$. In particular we have two functions $f_i : X \to \mathbb{R}$, $i = 1, 2$. Prove that f is continuous if and only if f_1 and f_2 are.

3) Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$. Use the inductive definition $x^n = x \cdot x^{n-1}$ and previous exercises to prove that f is continuous.

4) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$. Prove that

$$\int_a^b f = 0$$

implies $f(x) = 0$ for all $x \in [a, b]$.

5) Let (X, d) be a metric space. Let (x_n) and (y_n) be two Cauchy sequences in X. Prove that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$.

6) Let $K \subset \mathbb{R}$ be a set consisting of 0 and all $1/n$, $n = 1, 2, 3, \ldots$. Prove that K is compact directly using the definition, i.e. every open cover has a finite subcover.

7) Let $F_1 \supseteq F_2 \supseteq \ldots$ be a descending sequence of non-empty compact subsets. Prove that $\cap_{n=1}^{\infty} F_n$ is non-empty.

8) Let (X, d) be a metric space and f_n a sequence of continuous functions $f_n : X \to \mathbb{R}$ uniformly converging to f. Let x_n be a sequence of points in X such that $\lim_n x_n = x \in X$. Prove that $\lim_n f_n(x_n) = f(x)$.

9) A subset $\mathbb{R}^n$ is convex if for any two points $x, y \in C$, the segment $[x, y]$ is contained in C. Prove that C is connected.