MATH 3210-4, HW V
DUE WEDNESDAY NOV 13

1) Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x-y)^2$. Prove that f is constant.

2) Let f be a differentiable function defined in a neighborhood of x. Assume that $f''(x)$ exists. Use L'Hospital's rule to prove that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

3) (Fixed Point Theorem.) Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| \leq C$ for some $0 \leq C < 1$ and all x. A number x is a fixed point for f if $f(x) = x$. Prove that f cannot have two fixed points. Let x_1 be any real number, and define a sequence by $x_{n+1} = f(x_n)$. Prove that the sequence $\{x_n\}$ is Cauchy. (Hint: $|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|$.) Prove that the limit is a fixed point of f.

4) (Concavity.) Let $f : (\alpha, \beta) \to \mathbb{R}$ be twice differentiable function such that $f'' \geq 0$ on the interval. Let $c \in (\alpha, \beta)$ and let $g(x)$ be the linear function whose graph is the tangent line of the graph of f at c i.e. $g(x) = f(c) + f'(c)(x - c)$. Prove that $f(x) \geq g(x)$ for $x \in (\alpha, \beta)$.

5) (Newton Method.) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable function. Let $[a, b]$ be a closed interval such that $f(a) < 0$ and $f(b) > 0$, $f'(x) \geq \delta > 0$, and $f''(x) \geq 0$ for $x \in [a, b]$. Prove that there is unique $c \in (a, b)$ such that $f(c) = 0$. Define a sequence by $x_1 = b$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Prove that the sequence is decreasing and bounded from below by c, it has a limit. (Hint: interpret the sequence using the tangent line of the graph of f at x_n, and use the previous exercise.) Prove that the limit is c. Check that the conditions are satisfied for $f(x) = x^2 - 2$ and the interval $[1, 2]$. What is the limit of the sequence $\{x_n\}$? Compute x_n for $n = 1, 2, 3, 4$.

6) Consider the power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, i.e. the sequence whose n-th term is $(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$. Compute the radius of convergence of this series. Use the theorem of Taylor to prove that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ for every x. Use this series to find a rational number that approximates $\sin(1/2)$ with an error less than $1/10^3$.