1) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function such that \( f \geq 0 \). Show that \( f = 0 \) if and only if \( \int_a^b f(x) \, dx = 0 \).

2) Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = 1 \) if \( x \) is in the Cantor set and \( f(x) = 0 \) otherwise. Prove that \( f \) is Riemann integrable and compute its integral.

3) Let \( V = \mathcal{R}[a, b] \) the vector space of Riemann integrable functions on \( [a, b] \). Prove that 
\[
||f|| = \int_a^b |f(x)| \, dx
\]
defines a semi-norm on \( V \) i.e. (1) \( ||f|| \geq 0 \) (2) \( ||\lambda \cdot f|| = |\lambda| \cdot ||f|| \) (3) \( ||f + g|| \leq ||f|| + ||g|| \) where \( f, g \in V \) and \( \lambda \in \mathbb{R} \) (a norm satisfies \( ||f|| = 0 \) implies \( f = 0 \)). Give an example of a non-zero function \( f \in V \) such that \( ||f|| = 0 \). Let \( U = C[a, b] \) be the subspace of continuous functions on \( [a, b] \). Prove that \( ||f|| \) is a norm when restricted to \( U \).

4) Continuing with notation and setting of the previous exercise. Then
\[
d(f, g) = ||f - g||
\]
defines a semi-distance on \( V \) i.e. all axioms of the distance function are satisfied except there are \( f \neq g \) such that \( d(f, g) = 0 \). To get a metric space one needs to identify all \( f \) and \( g \) such that \( d(f, g) = 0 \). More precisely, we have an equivalence relation \( f \sim g \) if \( d(f, g) = 0 \) and then \( \bar{V} \), the set of equivalence classes, is a proper metric space. On the other hand, \( d \) is a proper distance function on \( U \) and \( U \) naturally embeds into \( \bar{V} \). Let

\[
f_n(x) = \begin{cases} 
-1 & \text{for } -1 \leq x \leq -1/n \\
nx & \text{for } -1/n \leq x \leq 1/n \\
1 & \text{for } 1/n \leq x \leq 1
\end{cases}
\]
be a sequence in \( U = C[-1, 1] \). Prove that \( f_n \) is a Cauchy sequence in \( U \). It does not have a limit point in \( U \), however it does in \( \bar{V} \). Find that limit.

5) Let \( f : [1, \infty) \to \mathbb{R} \) be a non-negative function such that \( f \) is integrable on the closed segment \([1, c]\) for every \( c \geq 1 \). One can define \( \int_1^\infty f(x) \, dx \) as the supremum of \( \int_1^c f(x) \, dx \) over all \( c \). Assume that \( f \) is monotone decreasing. Prove that \( \int_1^\infty f(x) \, dx \) is finite if and only if \( \sum_{n=1}^{\infty} f(n) \) converges. Apply this to prove that the series \( \sum_{n=1}^{\infty} n^{-s} \) for \( s > 0 \) is convergent if and only if \( s > 1 \).

6) Let \( (X, d) \) be a metric space and \( f_n \) a sequence of continuous functions \( f_n : X \to \mathbb{R} \) uniformly converging to \( f \). Let \( x_n \) be a sequence of points in \( X \) such that \( \lim_n x_n = x \in X \). Prove that \( \lim_n f_n(x_n) = f(x) \).