1) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( |f(x) - f(y)| \leq (x - y)^2 \). Prove that \( f \) is constant.

2) Let \( f \) be a differentiable function defined in a neighborhood of \( x \). Assume that \( f''(x) \) exists. Use L'Hospital's rule to prove that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x).
\]

3) (Fixed Point Theorem.) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( |f'(x)| \leq C \) for some \( 0 \leq C < 1 \) and all \( x \). A number \( x \) is a fixed point for \( f \) if \( f(x) = x \). Prove that \( f \) cannot have two fixed points.

Let \( x_1 \) be any real number, and define a sequence by
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]
Prove that the sequence \( \{x_n\} \) is Cauchy. (Hint: \( |x_{n+1} - x_n| \leq C|x_n - x_{n-1}| \).) Prove that the limit is a fixed point of \( f \).

4) (Concavity.) Let \( f : (\alpha, \beta) \rightarrow \mathbb{R} \) be twice differentiable function such that \( f'' \geq 0 \) on the interval. Let \( c \in (\alpha, \beta) \) and let \( g(x) \) be the linear function whose graph is the tangent line of the graph of \( f \) at \( c \) i.e. \( g(x) = f(c) + f'(c)(x - c) \). Prove that \( f(x) \geq g(x) \) for \( x \in (\alpha, \beta) \).

5) (Newton Method.) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be twice differentiable function. Let \( [a, b] \) be a closed interval such that \( f(a) < 0 \) and \( f(b) > 0 \), \( f'(x) \geq \delta > 0 \), and \( f''(x) \geq 0 \) for \( x \in [a, b] \). Prove that there is unique \( c \in (a, b) \) such that \( f(c) = 0 \). Define a sequence by \( x_1 = b \) and
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]
Prove that the sequence is decreasing and bounded from below by \( c \), it has a limit. (Hint: interpret the sequence using the tangent line of the graph of \( f \) at \( x_n \), and use the previous exercise.) Prove that the limit is \( c \). Check that the conditions are satisfied for \( f(x) = x^2 - 2 \) and the interval \([1, 2]\). What is the limit of the sequence \( \{x_n\} \)? Compute \( x_n \) for \( n = 1, 2, 3, 4 \).

6) Consider the power series \( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \), i.e. the sequence whose \( n \)-th term is \((-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \). Compute the radius of convergence of this series. Use the theorem of Taylor to prove that \( \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \) for every \( x \). Use this series to find a rational number that approximates \( \sin(1/2) \) with an error less than \( 1/10^3 \).