1) Let $f : X \rightarrow Y$ be a continuous map of metric spaces. Let $E \subset X$. Prove that $f(E) \subseteq f(E)$. Give an example where the inclusion is proper.

2) Let $F_1 \supseteq F_2 \supseteq \ldots$ be a descending sequence of non-empty compact sets in a metric space $X$. Prove that $\bigcap_{n=1}^{\infty} F_n$ is not empty. Give an example of a descending sequence of closed sets such that the intersection is empty.

3) Let $f : X \rightarrow Y$ be a uniformly continuous map of metric spaces. If $x_n$ is a Cauchy sequence in $X$, prove that $f(x_n)$ is a Cauchy sequence in $Y$. Give an example of a continuous function such that this fails.

4) Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Prove that there exists $x \in [0, 1]$ such that $f(x) = x$.

5) Let $X$ be a metric space, and fix a non-empty subset $E \subset X$. The distance from $x \in X$ to $E$ is defined by

$$d(x, E) = \inf_{y \in E} d(x, y).$$

Prove that $d(x, E) = 0$ if and only if $x \in \overline{E}$. Prove that

$$|d(x, E) - d(y, E)| \leq d(x, y)$$

i.e. $x \mapsto d(x, E)$ is a uniformly continuous function.