1) A complex number \( z \) is called algebraic if there exists integers \( a_0, a_1, \ldots, a_n \), such that 
\[ a_n z^n + \cdots + a_1 z + a_0 = 0. \]
Prove that the set of algebraic numbers is countable.

2) Prove that the following two \((X, d)\) are metric spaces:
   - \( X = \mathbb{R}^2 \) and \( d((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|) \).
   - \( X = \mathbb{Z} \) and \( d(x, x) = 0 \) or \( d(x, y) = \frac{1}{2^n} \), if \( x \neq y \), where \( 2^n \) is the largest power of 2 dividing \( x - y \).

3) Let \((X, d)\) be a metric space. The closed ball centered at \( x \) and of radius \( r > 0 \) is the set of \( y \in X \) such that \( d(x, y) \leq r \). Prove that the complement of the closed ball is an open set in \( X \).

4) Let \((X, d)\) be a metric space. Recall that the closure of \( E \subset X \) is \( \bar{E} \supseteq E \) obtained by adding limit points to \( E \). Since \( \bar{E} \) is larger than \( E \), it seems possible that \( \bar{E} \) has additional limit points, i.e. the closure does not give a closed set. Prove that
\[ \bar{E} = \bigcap_{E \subseteq F, F = \text{closed}} F \]
i.e. the intersection is taken over all closed sets \( F \) containing \( E \). In particular, \( \bar{E} \) is closed, why?

Let \( X \) be a metric or, more generally, a topological space. A collection of open sets \( \{V_\alpha\} \) in \( X \) is called a base for \( X \) if for any open set \( V \) and \( x \in V \) there exists a \( V_\alpha \) in the collection such that \( V_\alpha \) is contained in \( V \) and it contains \( x \). In particular, any open set can be written as a union of a subcollection of \( \{V_\alpha\} \). For example, if \( X \) is a metric space, then the collection of all open balls is a base for \( X \), by the definition of open sets. Topological spaces with a countable base are called separable.

5) Assume that a metric space \( X \) contains a countable subset \( X_0 \) such that the closure of \( X_0 \) is \( X \). Prove that the collection of balls centered at \( x \in X_0 \) and rational radii is a countable base for \( X \).

6) Prove that convex sets in \( \mathbb{R}^2 \) are connected, in the sense of definition 2.45.