
Volume 319 Number 8, April 15, 2008
ISSN 0021-8693
JOURNAL OF
ALGEBRA
Editor-in-Chief
Michel Broué
Editor of the section on Computational Algebra
Gerhard Hiss
Founding Editor
G. Higman, 1964–1984
Editor-in-Chief
W. Feit, 1985–2000
Editorial Board
Nicolás Andruskiewitsch
Luchezar L. Avramov
Eva Bayer-Fluckiger
Corrado de Concini
Steven Dale Cutkosky
Dihua Jiang
Masaki Kashiwara
E.I. Khukhro
Kazuhiko Kurano
Gus I. Lehrer
Martin Liebeck
Peter Littelmann
Laurent Moret-Bailly
Leonard L. Scott, Jr.
Vera Serganova
Aner Shalev
Ronald Solomon
J.T. Stafford
Gernot Stroth
Michel Van den Bergh
Efim Zelmanov
Section on Computational Algebra
Jon Carlson
John Cremona
Patrick Dehornoy
Harm Derksen
Meinolf Geck
Derek Holt
William M. Kantor
Reinhard Laubenbacher
Gunter Malle
Eamonn O'Brien
Bruno Salvy
Jean-Yves Thibon

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Algebra 319 (2008) 3244–3258
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Lifting of generic depth zero representations of classical groups

Gordan Savin

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

Received 9 June 2007
Available online 31 October 2007
Communicated by Dihua Jiang

Abstract

We describe how generic depth zero supercuspidal representations of classical groups lift to a general linear group. The main tool is a computation of reducibility points of certain parabolically induced representations.

Keywords: Classical groups; Supercuspidal representations; Lift

1. Introduction

Let G be a split classical group, either $\mathrm{SO}_{2n+1}(k)$ or $\mathrm{Sp}_{2n}(k)$, over a p-adic field k. We assume that p is odd. A goal of this paper is to describe how generic depth zero supercuspidal representations of G, studied by DeBacker and Reeder [DR], lift to a general linear group under the lift of Jiang and Soudry [JS] (for $\mathrm{SO}_{2n+1}(k)$) and under the lift of Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPS] (for $\mathrm{Sp}_{2n}(k)$).

Let W_k be the Weyl group of the local field and let $\mathcal{I} \supset \mathcal{I}^+$ be the inertia and the wild inertia subgroups of W_k. Let $G^*(\mathbb{C})$ be the dual group of G. Recall that a Langlands parameter is a homomorphism from $W_k \times \mathrm{SL}_2(\mathbb{C})$ to $G^*(\mathbb{C})$. A Langlands parameter φ is tame, regular, and discrete series if, respectively,

(1) φ is trivial on the wild inertia subgroup $\mathcal{I}^+$,

E-mail address: savin@math.utah.edu.

0021-8693/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2007.10.018

(2) the centralizer of $\varphi(\mathcal{I})$ is a maximal torus in $G^*(\mathbb{C})$,
(3) the centralizer of $\varphi(W_k)$ in $G^*(\mathbb{C})$ is finite.

Note that the second condition implies that φ is trivial on $\mathrm{SL}_2(\mathbb{C})$. Tame regular discrete series parameters can be explicitly described for classical groups. (See an excellent article of Gross and Reeder [GR].) Consider first the case $G = \mathrm{SO}_{2n+1}(k)$. The dual group of $\mathrm{SO}_{2n+1}(k)$ is $\mathrm{Sp}(V)$ where V is a $2n$-dimensional symplectic space over $\mathbb{C}$. A tame regular discrete series parameter for $\mathrm{SO}_{2n+1}(k)$ is a homomorphism

$$\varphi : W_k \to \mathrm{Sp}(V)$$

such that under the action of W_k the symplectic space V decomposes into irreducible summands

$$V = V_1 \oplus \cdots \oplus V_d$$

where each V_i is a non-degenerate symplectic subspace of V of dimension $2n_i$. Moreover, if φ_i denotes the representation of W_k on V_i, then $\varphi_i \cong \varphi_j$ if and only if $i = j$. Irreducible tame symplectic representations of W_k can be explicitly described and to every tame regular discrete series parameter one can attach a depth zero supercuspidal generic representation Σ of $\mathrm{SO}_{2n+1}(k)$ by picking a hyperspecial maximal compact subgroup of $\mathrm{SO}_{2n+1}(k)$. (See [DR] and [GR] how to attach a whole packet of representations to φ.) The representation Σ depends on the choice of the hyperspecial maximal compact subgroup, however, this choice is not important to us. These representations exhaust all generic depth zero supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$.

Recall now that to every irreducible summand φ_i of φ one can attach, via the local Langlands correspondence, a depth zero self-dual supercuspidal representation Π_{φ_i} of $\mathrm{GL}_{2n_i}(k)$. Let Π be a depth zero self-dual supercuspidal representation of $\mathrm{GL}_{2m}(k)$. Consider the generalized principal series

$$\mathrm{I}(s, \Pi \otimes \Sigma)$$

of $\mathrm{SO}_{4m+2n+1}(k)$ obtained by inducing $\Pi \otimes \Sigma$ from a maximal parabolic subgroup $P = MN$ with the Levi factor $M = \mathrm{GL}_{2m} \times \mathrm{SO}_{2n+1}$. The induction is normalized as in [Sh]. Our first result is:

Theorem 1.1. *Let Σ be a depth zero generic supercuspidal representation of $\mathrm{SO}_{2n+1}(k)$ corresponding, in the sense of DeBacker and Reeder, to a tame regular discrete series parameter $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_d$. Let Π be a depth zero self-dual supercuspidal representation of $\mathrm{GL}_{2m}(k)$. Then the generalized principal series $\mathrm{I}(s, \Pi \otimes \Sigma)$ reduces at $s = 1$ if and only if $\Pi \cong \Pi_{\varphi_i}$ for some i.*

On the other hand, rank one reducibilities are determined by the Jiang–Soudry lift Σ to $\mathrm{GL}_{2n}(k)$ and vice versa. (See a discussion on this topic in [J, Chapter 3].) Thus, a consequence of Theorem 1.1 is that the Jiang–Soudry lift of Σ to $\mathrm{GL}_{2n}(k)$ is

$$\mathrm{Ind}(\Pi_{\varphi_1} \otimes \cdots \otimes \Pi_{\varphi_d}),$$

where the induction is from a parabolic subgroup with Levi factor $\mathrm{GL}_{2n_1} \times \cdots \times \mathrm{GL}_{2n_d}$. The induction is normalized so that the resulting representation is tempered. In particular, this shows that the parametrizations of Debacker–Reeder and Jiang–Soudry coincide for generic supercuspidal representations of depth zero.

We also have similar results for $\mathrm{Sp}_{2n}(k)$. The dual group of $\mathrm{Sp}_{2n}(k)$ is $\mathrm{SO}(U)$ where U is a $2n+1$-dimensional orthogonal space over $\mathbb{C}$. The main difference in this case is that there are four one-dimensional orthogonal parameters: the trivial character and three non-trivial quadratic characters of $W_k^{ab} \cong k^\times$. Any other tame irreducible orthogonal representation of W_k is even-dimensional and its determinant is χ, the unique unramified quadratic character. A tame regular discrete series parameter for $\mathrm{Sp}_{2n}(k)$ is a homomorphism

$$\phi : W_k \to \mathrm{SO}(U)$$

such that under the action of W_k the orthogonal space U decomposes into irreducible summands

$$U = U_1 \oplus \cdots \oplus U_d \oplus \chi^d$$

where each U_i is a non-degenerate orthogonal subspace of U of dimension $2n_i$. Moreover, if ϕ_i denotes the representation of W_k on U_i, then $\phi_i \cong \phi_j$ if and only if $i = j$. Again, tame irreducible orthogonal parameters can be explicitly described and to every tame regular discrete series parameter ϕ one can attach a depth zero supercuspidal generic representation Σ of $\mathrm{Sp}_{2n}(k)$ by picking a hyperspecial maximal compact subgroup of $\mathrm{Sp}_{2n}(k)$, as well as a depth zero, self-dual supercuspidal representation Π_{ϕ_i} of $\mathrm{GL}_{2n_i}(k)$, for every even-dimensional summand ϕ_i of ϕ. Let Π be a depth zero self-dual supercuspidal representation of $\mathrm{GL}_{2m}(k)$. Consider the generalized principal series

$$\mathrm{I}(s, \Pi \otimes \Sigma)$$

of $\mathrm{Sp}_{4m+2n}(k)$ obtained by inducing $\Pi \otimes \Sigma$ from a maximal parabolic subgroup $P = MN$ with the Levi factor $M = \mathrm{GL}_{2m} \times \mathrm{Sp}_{2n}$. Again, the induction is normalized as in [Sh].

Theorem 1.2. *Let Σ be a depth zero generic supercuspidal representation of $\mathrm{Sp}_{2n}(k)$ corresponding, in the sense of DeBacker and Reeder, to a tame regular discrete series parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_d \oplus \chi^d$. Let Π be a depth zero self-dual supercuspidal representation of $\mathrm{GL}_{2m}(k)$. Then the generalized principal series $\mathrm{I}(s, \Pi \otimes \Sigma)$ reduces at $s = 1$ if and only if $\Pi \cong \Pi_{\phi_i}$ for some i.*

A consequence of this result is that the lift of Σ to $\mathrm{GL}_{2n+1}(k)$, in the sense of Theorem 7.3 in [CKPS], is

$$\mathrm{Ind}\big(\Pi_{\phi_1} \otimes \cdots \otimes \Pi_{\phi_d} \otimes \chi^d\big).$$

Perhaps we need to say a word or two about depth zero representations of $\mathrm{Sp}_{2n}(k)$ not covered by the above theorem. The simplest tame non-regular discrete series parameter is 3-dimensional. It is a sum of three non-trivial quadratic characters of W_k^{ab}. This is a tame discrete series parameter for $\mathrm{SL}_2(k)$. The L-packet consists of 4 representations. The group $\mathrm{SL}_2(k)$ has two conjugacy classes of hyperspecial maximal compact subgroups. Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be representatives of the two classes. The four representations in the packet are constructed by taking two cuspidal representations of $\mathrm{SL}_2(\mathbb{F}_q)$ of dimension $(q-1)/2$ (these are cuspidal components of two oscillator representations) and inducing them, via $\mathcal{K}_0$ and $\mathcal{K}_1$ to $\mathrm{SL}_2(k)$. The two cuspidal representations of $\mathrm{SL}_2(\mathbb{F}_q)$ are obtained from a Deligne–Lusztig character $R_{T,\theta}$ with θ not in general position, see the definition on p. 219 in Carter's book [Ca]. In essence, the restriction to regular parameters allows us to avoid certain annoyances associated with groups with disconnected center.

Our method is based on a simple observation (see Section 4) that the generalized principal series $\mathrm{I}(s, \Pi \otimes \Sigma)$ reduces at $s = 1$ if and only if a certain Hecke algebra of type $\widetilde{A}_1$ has unequal parameters. This information can be easily read off of a work of Lusztig [Lu]. We do not determine precisely the Hecke algebra.

2. Tame self-dual parameters of Weyl group

Let $\Omega_{p'}$ denote the set of all complex roots of unity of order prime to p. Let $\mathbb{F}_q$ be the residual field of k and $\overline{\mathbb{F}}_q$ its algebraic closure. We fix once and for all an identification of $\overline{\mathbb{F}}_q^\times$ and $\Omega_{p'}$ [Ca, Proposition 3.1.3]. In particular, for every n, we have a primitive root ζ_n in $\mathbb{F}_{q^n}^\times$ that corresponds to $e^{2\pi i/(q^n-1)}$ under this identification. (The actual choice of identification is not important. As the reeder will soon see, the identification is simply used as a convenience to "write down" tame parameters of the Weyl group.) The Frobenius acts on $\Omega_{p'}$ by

$$F(\tau) = \tau^q$$

for every τ in $\Omega_{p'}$. Note that all F-orbits are finite. These orbits play a key role in the description of tame parameters.

Lemma 2.1. *Let τ be a root of* 1 *different from* ± 1. *Assume that the F-orbit of τ has n different elements*:

$$\tau, \tau^q, \ldots, \tau^{q^{n-1}}.$$

If τ^{-1} is on this list, that is, if $\tau^{-1} = \tau^{q^m}$ for some $m < n$ then $n = 2m$.

Proof. First of all, note that $0 < m$ since $\tau \neq \pm 1$. Raising $\tau^{-1} = \tau^{q^m}$ to the q^mth power gives $\tau = \tau^{q^{2m}}$. Since $\tau = \tau^k$ if and only if k is a multiple of n, and $0 < 2m < 2n$, it follows that $n = 2m$, as claimed. $\square$

We are now ready to define irreducible tame self-dual parameters. Let ϖ be a uniformizing element in k. Let K be the unique unramified extension of k of degree $2m$. Then

$$K^\times = \langle \varpi \rangle \times \mathbb{F}_{q^{2m}}^\times \times U_1$$

where U_1 is the maximal pro p-subgroup of $K^\times$. A character of $K^\times$ is called tame if it is trivial on U_1. Let ζ_{2m} be the primitive root in $\mathbb{F}_{q^{2m}}^\times$ defined above. Pick τ, a complex root of 1 such that the F-orbit $\tau, \tau^q, \ldots$ has precisely $2m$ distinct elements and $\tau^{q^m} = \tau^{-1}$. (For example, τ can be picked a primitive root of order $q^m + 1$.) Then τ defines two tame characters—denoted by the same symbol—η of $K^\times$ by

$$\begin{cases} \eta(\zeta_{2m}) = \tau, \\ \eta(\varpi) = -1 \text{ or } 1. \end{cases}$$

Let W_k and W_K be the local Weyl groups of k and K. Recall that

$$W_k/W_K \cong \mathrm{Gal}(K/k) \cong \mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q).$$

Via the local class field theory we have an identification $W_K^{ab} \cong K^\times$. Note that $\eta \circ F^i \neq \eta$ and $\eta \circ F^m = \bar{\eta}$. In particular, the character η defines two $2m$-dimensional irreducible and self-dual representations

$$\begin{cases} \varphi(\tau) = \mathrm{Ind}_{W_K}^{W_k}(\eta) & \text{if } \eta(\varpi) = -1, \\ \phi(\tau) = \mathrm{Ind}_{W_K}^{W_k}(\eta) & \text{if } \eta(\varpi) = 1 \end{cases}$$

of W_k. We record (see [Moy]) that φ is symplectic and ϕ is orthogonal. Moreover, the determinant of ϕ is χ, the unique unramified quadratic character of $W_k^{ab} \cong k^\times$. The two parameters, φ and ϕ, are obtained one from another by twisting by an unramified character of $W_k^{ab} \cong k^\times$ of order $4m$.

Summarizing, irreducible $2m$-dimensional symplectic and orthogonal parameters are parameterized by F-obits $\tau, \tau^q, \ldots$ in $\Omega_{p'}$ of order $2m$ such that $\tau^{q^m} = \tau^{-1}$. It follows that a tame regular discrete series parameter, symplectic or orthogonal, is given by a sequence $\tau_1, \ldots, \tau_d$ of roots in $\Omega_{p'}$ belonging to different F-orbits. We write these two parameters, symplectic and orthogonal, respectively, by

$$\begin{cases} \varphi(\tau_1, \ldots, \tau_d), \\ \phi(\tau_1, \ldots, \tau_d). \end{cases}$$

3. Reductive groups over finite fields

Let $\varphi(\tau_1, \ldots, \tau_d)$ or $\phi(\tau_1, \ldots, \tau_d)$ be a tame regular discrete series parameter of $\mathrm{SO}_{2n+1}(k)$ or $\mathrm{Sp}_{2n}(k)$, respectively. In order to describe the generic supercuspidal representation Σ attached to this parameter we need to recall some results of Deligne and Lusztig.

Let (G, F) denote a pair where G is a split group GL_n, SO_{2n+1} or Sp_{2n} defined over the finite field $\mathbb{F}_q$ and F is the action of Frobenius on $G(\bar{\mathbb{F}}_q)$. Regular cuspidal representations of G are parameterized by pairs (T, θ) where T is an F-stable elliptic torus of G and θ is a regular character of T. The character θ can be viewed as a regular element in a dual F^*-stable torus T^* in the dual group (G^*, F^*). In the cases of our interest, we record

G	GL_n	SO_{2n+1}	Sp_{2n}
G^*	GL_n	Sp_{2n}	SO_{2n+1}

We now proceed to describe the conjugacy classes of F-stable elliptic tori in G^*. We first start with GL_n. If T^* is a maximal torus in G then, over the algebraic closure, $T^*(\bar{\mathbb{F}}_q) \cong X \otimes \bar{\mathbb{F}}_q^\times$ where X is the co-character lattice. In this case we have a natural identification

$$X = \mathbb{Z}^n = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$$

so that the action of the absolute Weyl group A_n on X is by permutations of vectors $e_1, \ldots, e_n$. Any element in T is represented by an n-tuple

$$(x_1, \ldots, x_n) = e_1 \otimes x_1 + \cdots + e_n \otimes x_n.$$

There is only one class of elliptic tori, given by the Coxeter torus. Recall that the Coxeter element, or rather the Coxeter conjugacy class, is given as a product of simple reflections. For GL_n this is a cycle of order n. The Frobenius action on T is obtained by twisting the usual Frobenius action on $\overline{\mathbb{F}}_q$ by the Coxeter element of the Weyl group. It follows that F^*-action on the elliptic T^* is given by

$$F^*(x_1, x_2, \dots, x_n) = \left(x_n^q, x_1^q, \dots, x_{n-1}^q\right).$$

A cuspidal parameter for $G = \mathrm{GL}_n$ is given by a regular element in the Coxeter torus T^*:

$$\theta = \left(\tau, \tau^q, \dots, \tau^{q^{n-1}}\right).$$

Let π be the irreducible Deligne–Lusztig character $\pm R_{T,\theta}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ attached to θ.

If $G^* = \mathrm{Sp}_{2n}$ or SO_{2n+1} then there is more then one class of F^*-stable elliptic tori, but they can be all easily understood in terms of the Coxeter tori. Let Φ be the absolute root system of G^*. We can realize Φ in the standard fashion so that the simple roots are

$$\alpha_1 = e_1 - e_2, \qquad \dots, \qquad \alpha_{n-1} = e_{n-1} - e_n$$

and

$$\alpha_n = \begin{cases} e_n & \text{if } \Phi = B_n, \\ 2e_n & \text{if } \Phi = C_n. \end{cases}$$

Again, for any maximal torus T^*, we have $T^*(\overline{\mathbb{F}}_q) \cong X \otimes \overline{\mathbb{F}}_q^\times$ where X is spanned by $e_1, \dots, e_n$. If T^* is a Coxeter then the action of F^* is given by

$$F^*(x_1, x_2, \dots, x_n) = \left(x_n^{-q}, x_1^q, \dots, x_{n-1}^q\right).$$

A general elliptic torus is a product of Coxeter tori $T_1^* \times \dots \times T_d^*$ corresponding to a decomposition

$$X = X_1 \oplus \dots \oplus X_d$$

where each X_i is spanned by a subset of $e_1, \dots, e_n$.

Now, to every tame, regular parameter (symplectic or orthogonal) given by roots $\tau_1, \dots, \tau_d$ we can attach a regular element

$$\theta = \theta_1 \times \dots \times \theta_d = \left(\tau_1, \dots, \tau_1^{q^{n_1-1}}, \dots, \tau_d, \dots, \tau_d^{q^{n_d-1}}\right)$$

in the elliptic torus $T_1^* \times \dots \times T_d^*$. (Note that $\tau_i^{q^{n_i}} = \tau_i^{-1}$ holds if the parameter is F-invariant.) Note a slight difference between the two cases. If $G^* = \mathrm{SO}_{2n+1}$ then θ is regular even if $\tau_i = -1$ for one i. In that case, however, θ is not in a general position. We assume that $\tau_i \neq \pm 1$ throughout this paper. In particular, θ is in a general position and the corresponding Deligne–Lusztig character

$$\sigma = \pm R_{T,\theta}$$

of $G(\mathbb{F}_q)$ is irreducible. Let Σ be the representation of $G(k)$ compactly induced from σ, understood as a representation of a hyperspecial maximal subgroup of $G(k)$. This representation is easily seen to be generic, see Lemma 6.1.1 in [DR].

4. Rank one reducibilities

Let Σ be a depth zero generic supercuspidal representation of $\mathrm{SO}_{2n+1}(k)$ attached to a tame parameter $\varphi(\tau_1, \dots, \tau_d)$ as in the previous section.

Let $\varphi(\tau)$ and $\phi(\tau)$ be the tame irreducible symplectic and orthogonal parameters, respectively, of dimension $2m$ attached to the same complex root τ such that $\tau^{q^m} = \tau^{-1}$. The determinant of $\phi(\tau)$ is the unique unramified quadratic character χ of $W_k^{ab} \cong k^\times$. The two parameters define two depth zero self-dual supercuspidal representations of $\mathrm{GL}_{2m}(k)$ as follows. Let

$$\theta = \left(\tau, \dots, \tau^{q^{m-1}}, \tau^{-1}, \dots, \tau^{-q^{m-1}}\right).$$

It is a parameter of a self-dual cuspidal representation π of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ with the trivial central character. Let R be the maximal order in k. The self-dual representations attached to $\varphi(\tau)$ and $\phi(\tau)$ via the local Langlands correspondence are

$$\begin{cases} \Pi_{\varphi(\tau)} = \mathrm{ind}_{k^\times \mathrm{GL}_{2m}(R)}^{\mathrm{GL}_{2m}(k)}(1 \otimes \pi), \\ \Pi_{\phi(\tau)} = \mathrm{ind}_{k^\times \mathrm{GL}_{2m}(R)}^{\mathrm{GL}_{2m}(k)}(\chi \otimes \pi). \end{cases}$$

This follows from a result of Henniart in [He1]. He shows there that the functorial lift of the character η of $K^\times$ to $\mathrm{GL}_{2m}(k)$, obtained by the trace formula, coincides with Howe's construction of supercuspidal representations of $\mathrm{GL}_{2m}(k)$ up to a twist by the unramified character of $K^\times$ of order 2. However, our choice of central characters for $\Pi_{\varphi(\tau)}$ and $\Pi_{\phi(\tau)}$ is made to account for the twist.

We note that $\Pi_{\varphi(\tau)}$ is a twist of $\Pi_{\phi(\tau)}$ by an unramified character of order $4m$. By a result of Henniart [He2],

$$\begin{cases} L\left(0, \wedge^2, \Pi_{\varphi(\tau)}\right) = \infty, \\ L\left(0, \mathrm{Sym}^2, \Pi_{\phi(\tau)}\right) = \infty \end{cases}$$

holds for Shahidi's exterior and symmetric square L-functions. Indeed, Henniart shows that the local Langlands correspondence preserves exterior and symmetric square L-functions. The statement about poles on the Galois side is obvious. Alternatively, we can use Theorem 7.6 in [KM]. That result is equivalent to the statement on the poles of Shahidi's L-functions.

We have come to the main point of this paper. Let $\mathrm{I}(s, \Pi_{\varphi(\tau)} \otimes \Sigma)$ and $\mathrm{I}(s, \Pi_{\phi(\tau)} \otimes \Sigma)$ be the generalized principal series representations of $\mathrm{SO}_{4m+2n+1}(k)$ obtained by inducing from a maximal parabolic $P = MN$, as in the introduction. Points of reducibility for these principal series representations are given in terms of poles of Shahidi's L-functions. (See Section 3 in [Mu] for a nice introduction to the connection of L-parameters and points of reducibility.) Since $L(0, \mathrm{Sym}^2, \Pi_{\phi(\tau)}) = \infty$, the generalized principal series $\mathrm{I}(s, \Pi_{\phi(\tau)} \otimes \Sigma)$ reduces at $s = 1/2$, that is, we have a complementary series here. On the other hand, the generalized principal series $\mathrm{I}(s, \Pi_{\varphi(\tau)} \otimes \Sigma)$ either reduces at $s = 0$, in which case there is no complementary series, or it reduces at $s = 1$. We would like to understand which of the two possibilities occurs for a given τ. Here is an idea. Since $\Pi_{\phi(\tau)}$ is a twist of $\Pi_{\varphi(\tau)}$ by an unramified character of order $4m$ the two families of principal series representations belong to the same Bernstein's component. Irreducible representations in this component are parameterized by representations of a Hecke algebra of type $\widetilde{A}_1$ [Mor]. Matsumoto showed in [Ma, p. 47] that a Hecke algebra of type $\widetilde{A}_1$ has two complementary series of different length if and only if it has unequal parameters. In particular, in order to show that a reducibility happens for $s = 1$, it suffices to show that the parameters of the Hecke algebra in question are unequal! This can be easily read off of a work of Lusztig [Lu] and it will be done in the next section. More precisely, we shall show that one parameter is a cube of the other. If that is the case, then Matsumoto's computation shows that the length of one complementary series is twice the length of the other and this is consistent with Shahidi's results.

The same approach works for Σ, a generic supercuspidal representation of $\mathrm{Sp}_{2n}(k)$, attached to a tame parameter $\phi(\tau_1, \ldots, \tau_d)$ in general position.

5. Hecke algebras

Let $\rho = \rho(\tau)$ be an irreducible tame symplectic or orthogonal parameter of dimension $2m$ constructed by means of τ, a complex root of 1. Let Π be the corresponding depth zero self-dual representation of $\mathrm{GL}_{2m}(k)$. Let Σ be a depth zero generic supercuspidal representation of $\mathrm{SO}_{2n+1}(k)$ or $\mathrm{Sp}_{2n}(k)$ attached to a parameter constructed by means of complex roots $\tau_1, \ldots, \tau_d$. Let G denote the split group $\mathrm{SO}_{4m+2n+1}$ or Sp_{4m+2n}. We shall now describe the parameters of the Hecke algebra of Bernstein's component containing representations induced from $\Pi \otimes \Sigma$, via the maximal parabolic group $P = MN$.

We realize the root system Φ of G in a standard fashion in $\mathbb{Z}^{2m+n}$. Let $\alpha_1, \ldots, \alpha_{2m+n}$ be the simple roots where α_{2m+n} is the unique short or long simple root. Note that the maximal parabolic P corresponds to the simple root α_{2m}. We extend the Dynkin diagram by adding the lowest root

$$\alpha_0 = -\begin{cases} e_1 + e_2 & \text{if } \Phi = B_{2m+n}, \\ 2e_1 & \text{if } \Phi = C_{2m+n}. \end{cases}$$

Let $\mathcal{K}_i$ be the maximal compact subgroup of $G(k)$ corresponding to the root α_i in the extended Dynkin diagram. Of interest to us are $\mathcal{K}_0$, a hyperspecial maximal compact, and $\mathcal{K}_{2m}$. Consider the intersection

$$\mathcal{P} = \mathcal{K}_0 \cap \mathcal{K}_{2m}.$$

Notice that $\mathcal{P} \cap M$ is a hyperspecial maximal compact of M, the Levi factor of P.

Recall that the complex roots τ and $\tau_1, \ldots, \tau_d$ define a pair of cuspidal representations π and σ of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ and of $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$ or $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, respectively. Bernstein's component of generalized principal series representations $\mathrm{I}(s, \Pi \otimes \Sigma)$ consists of representations of $G(k)$ generated by the $\mathcal{P}$-type

$$\delta = \pi \otimes \sigma.$$

The corresponding Hecke algebra H is of type $\widetilde{A}_1$ and it can be determined following Theorems 4.15 and 7.12 in [Mor]. (A detailed discussion, in the case of Siegel maximal parabolic subgroup, is given in Proposition 6.1 in [KM].) We summarize what we need here: The algebra

H is generated by two elements T_i supported in $\mathcal{K}_i$, for $i=0$ and $2m$. More precisely, let G_i be the quotient of $\mathcal{K}_i$ by its pro-unipotent radical. Let $P_i = M_i N_i$ be the image of $\mathcal{P}$ in G_i. Consider the induced representation $V_i = \mathrm{Ind}_{P_i}^{G_i}(\delta)$. Let

$$H_i = \mathrm{End}_{G_i}(V_i)$$

be the algebra of endomorphism of V_i. Note that H_i is contained in H. By a result of Howlett and Lehrer [HL] H_i is a Hecke algebra of type A_1. It is generated by an element T_i satisfying a quadratic relation

$$T_i^2 = (p_i - 1)T_i + p_i$$

where $p_i \geqslant 1$ is the quotient of the degrees of two irreducible summands of V_i. (Note that there are two irreducible summands since the algebra of endomorphisms is two-dimensional.)

Lusztig in [Lu, Theorem 8.6] gives a way to compute the numbers p_i. The reminder of this section is devoted to calculating p_i's on a case by case basis.

Case $i=2m$. In this case the reductive quotient G_{2m} is equal to

$$\begin{cases} \mathrm{SO}_{4m} \times \mathrm{SO}_{2n+1} & \text{if } \Phi = B_{2m+n}, \\ \mathrm{Sp}_{4m} \times \mathrm{Sp}_{2n} & \text{if } \Phi = C_{2m+n}. \end{cases}$$

In particular, the business of calculating p_{2m} involves only inducing π from $\mathrm{GL}_{2m}(\mathbb{F}_q)$ to $\mathrm{SO}_{4m}(\mathbb{F}_q)$ or $\mathrm{Sp}_{4m}(\mathbb{F}_q)$. This calculation was done in [KM] and, we record,

$$p_{2m} = q^m.$$

Case $i=0$. In this case K_0 is a hyperspecial maximal compact subgroup of G. Let G_0^* and M_0^* be the dual groups of G_0 and M_0. Note that M_0^* is equal to

$$\begin{cases} \mathrm{GL}_{2m} \times \mathrm{Sp}_{2n} & \text{if } \Phi = B_{2m+n}, \\ \mathrm{GL}_{2m} \times \mathrm{SO}_{2n+1} & \text{if } \Phi = C_{2m+n}. \end{cases}$$

In M_0^* we have an F^*-stable elliptic torus

$$T_0^* = T^* \times T_1^* \times \cdots \times T_d^* \cong \left(\overline{\mathbb{F}}_q^{\times}\right)^{2m+n}$$

where T^* is the F^*-stable elliptic torus in GL_{2m} and $T_1^* \times \cdots \times T_d^*$ is the F^*-stable elliptic torus in Sp_{2n} or in SO_{2n+1}, as described in Section 3. The Deligne–Lusztig parameter of δ is a regular element

$$\theta_0 = \left(\tau, \ldots, \tau^{q^{m-1}}, \tau^{-1}, \ldots, \tau^{-q^{m-1}}, \tau_1, \ldots, \tau_1^{q^{n_1-1}}, \ldots, \tau_d, \ldots, \tau_d^{q^{n_d-1}}\right)$$

in general position. Let W be the centralizer of θ_0 in the absolute Weyl group. There are two cases.

Case τ is in the F-orbit of some τ_i. Without loss of generality we can assume that $i = 1$, $n_1 = m$ and $\tau_1 = \tau$. Then the centralizer W of θ_0 is generated by reflections corresponding to roots

$$I = \{e_j - e_{2m+j} \text{ and } e_{m+j} + e_{2m+j} \mid j = 1, \dots, m\}.$$

These $2m$ roots span, as simple roots, a system of type A_2^m. It follows that

$$W \cong W_{A_2}^m,$$

m copies of the Weyl group of type A_2. Note also that F^* acts transitively on roots in I. This observation is in accordance with Theorem 8.6 in [Lu]. That is, the Hecke algebra of the induced representation of the finite group G_0 is 2-dimensional, the element T_0 corresponds to the unique F^*-orbit on S and p_0 is equal to the dimension of the Steinberg representation of W:

$$p_0 = q^{3m}.$$

We need to say a couple of words how the exponent $3m$ is computed in the case at hand. The formula (8.2.3) in [Lu], applied to the pair $(W_I, W') = (W, 1)$, says that

$$p_0 = q^{a_{E''} - a_{E'}}$$

where $E' = J_1^W(1)$ and $E'' = E' \otimes \text{sign}$ are two representations of W. We recall that $J_1^W(1)$ is a sum of all irreducible representations $\mathcal{W}$ of W such that $a_{\mathcal{W}} = 0$. A nice exposition of the a-function is given in the Appendix in [KM]. This is what we need here: Recall that the group W_{A_2} has three irreducible representations: 1, ρ and sign. The a-function $\mathcal{W} \mapsto a_{\mathcal{W}}$ for irreducible representations of W_{A_2} is given by

$\mathcal{W}$	1	ρ	sign
$a_{\mathcal{W}}$	0	1	3

Since $W = W_{A_2}^m$, any irreducible representation $\mathcal{W}$ of W is a tensor product $\mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_m$ of irreducible representations of W_{A_2}. Since (see the Appendix in [KM])

$$a_{\mathcal{W}} = a_{\mathcal{W}_1} + \cdots + a_{\mathcal{W}_m}$$

it easily follows that E' is the trivial representation and E'' the sign representation of W. In particular, $a_{E''} - a_{E'} = 3m - 0$, as desired. This completes the calculation of the parameter p_0, and shows that the Hecke algebra has unequal parameters, in this case, as desired.

Case τ is not in the F-orbit of any τ_i. In this case the centralizer W of θ_0 is generated by reflections

$$I = \{e_j + e_{m+j} \mid j = 1, \dots, m\}.$$

These roots span a root system of type A_1^m. It follows that

$$W \cong W_{A_1}^m,$$

m copies of the Weyl group of type A_1. Again F^* acts transitively on the roots in I. The Steinberg representation of W has the dimension q^m, so in this case,

$$p_0 = q^m$$

and we have equal parameters as we should. In particular, we have proved Theorems 1.1 and 1.2 with the following caveat:

The result of Lusztig is subject to an assumption that G_0 has connected center. This holds for SO_{2n+1} but not for Sp_{2n}. In order to remedy this, let $G'_0 = \mathrm{GSp}_{2n}$. Then we have a surjective map of dual groups

$$\psi : G_0'^* \to G_0^* = \mathrm{SO}_{2n+1}.$$

Let θ'_0 be an element in G'^* such that $\psi(\theta'_0) = \theta_0$. Then θ'_0 defines an irreducible cuspidal representation δ' of M'_0. The restriction of δ' to M_0 is isomorphic to δ. Indeed, since θ_0 is in general position, δ is given by an irreducible Deligne–Lusztig character. The dimensions of δ and δ' are equal, by Theorem 7.5.1 in [Ca].

Next, note that the centralizer of θ'_0 in the Weyl group is isomorphic to W, the centralizer of θ_0. Indeed, since $\psi(\theta'_0) = \theta_0$, the centralizer of θ'_0 is included in the centralizer of θ_0. On the other hand, note that the centralizer of θ_0 is generated by reflections about the roots α in S, and $\alpha(\theta_0) = 1$, for every α in S. Since

$$\alpha\big(\theta'_0\big) = \alpha(\theta_0),$$

the reflection about α centralizes θ'_0, as well. It follows that the pair (W, S) is the same for both, θ_0 and θ'_0. The result of Luzstig can now be applied, and we have

$$\mathrm{Ind}_{P'_0}^{G'_0}(\delta') = \delta'_1 \oplus \delta'_2,$$

where the quotient of the two degrees on the right is q^{3m} or q^m, depending whether τ is in the F-orbit of some τ_i or not. Since the restriction of $\mathrm{Ind}_{P'_0}^{G'_0}(\delta')$ is isomorphic to

$$\mathrm{Ind}_{P'_0}^{G'_0}(\delta) = \delta_1 \oplus \delta_2,$$

the ratio of the degrees of δ_1 and δ_2 is also q^{3m} and q^m in the two cases, respectively.

In the above argument we used that θ_0 is in general position. Otherwise the centralizer of θ_0 might include a reflection about a root α such that $\alpha(\theta_0) = -1$. Then this reflection would not be in the centralizer of θ'_0. This sort of situation happens when we want to extend a cuspidal representation of $\mathrm{SL}_2(\mathbb{F}_q)$ of dimension $(q-1)/2$ to $\mathrm{GL}_2(\mathbb{F}_q)$.

6. Lifting to GL_{2n} and GL_{2n+1}

Let Σ be a generic supercuspidal representation of the split group $\mathrm{SO}_{2n+1}(k)$, not necessarily of depth zero. Jiang and Soudry (Theorem 6.1 in [JS]) defined a lift of Σ as the unique representation of Π_Σ to $\mathrm{GL}_{2n}(k)$ such that

$$\begin{cases} L(s,\Sigma\times\pi)=L(s,\Pi_\Sigma\times\pi),\\ \epsilon(s,\Sigma\times\pi)=\epsilon(s,\Pi_\Sigma\times\pi)\end{cases}$$

for any representation π of any $\mathrm{GL}_m(k)$. We note that this result appears as a part of Theorem 7.3 in [CKPS]. Moreover, Theorem 7.3 in [CKPS] contains a generalization to other classical groups that we shall need.

Let $\varphi=\varphi_1\oplus\cdots\oplus\varphi_d$ be a regular tame discrete series parameter for $\mathrm{SO}_{2n+1}(k)$. Recall that φ defines a cuspidal and generic representation Σ of $\mathrm{SO}_{2n+1}(k)$ and a self-dual depth zero supercuspidal representation Π_{φ_i} of $\mathrm{GL}_{2n_i}(k)$ for every $i=1,\dots,d$.

Theorem 6.1. *Let Σ be a depth zero generic supercuspidal representation of* $\mathrm{SO}_{2n+1}(k)$ *corresponding, in the sense of DeBacker and Reeder, to a tame regular discrete series symplectic parameter $\varphi=\varphi_1\oplus\cdots\oplus\varphi_d$. Then*

$$\mathrm{Ind}(\Pi_{\varphi_1}\otimes\cdots\otimes\Pi_{\varphi_d})$$

is the lift Π_Σ of Σ to $\mathrm{GL}_{2n}(k)$*, in the sense of Jiang and Soudry as explained above. In particular,*

$$\begin{cases} L(s,\Sigma\times\pi)=\displaystyle\prod_{i=1}^{d}L(s,\Pi_{\varphi_i}\times\pi),\\ \epsilon(s,\Sigma\times\pi)=\displaystyle\prod_{i=1}^{d}\epsilon(s,\Pi_{\varphi_i}\times\pi)\end{cases}$$

holds for any representation π of any $\mathrm{GL}_m(k)$.

Proof. Jiang and Soudry show that there exist self-dual and mutually non-isomorphic cuspidal representations π_i of $\mathrm{GL}_{m_i}(k)$ $(m_1+\cdots+m_r=2n)$ such that Σ lifts to

$$\Pi_\Sigma=\mathrm{Ind}(\pi_1\otimes\cdots\otimes\pi_r).$$

Since $L(s,\Sigma\times\pi)=L(s,\Pi_\Sigma\times\pi)$, for any representation π of any $\mathrm{GL}_m(k)$, by multiplicativity property of L-functions we have

$$L(s,\Sigma\times\pi)=L(s,\Pi_\Sigma\times\pi)=\prod_{i=1}^{r}L(s,\pi_i\times\pi).$$

Since $L(0,\pi_i\times\pi_i)=\infty$ for any i it follows, by taking $\pi=\pi_i$, that

$$L(0,\Sigma\times\pi_i)=\infty$$

for all i. By a result of Moeglin (the main inequality in [Moe]) there are no other self-dual representations π such that $L(0, \Sigma \times \pi) = \infty$. By Theorem 1.1 the generalized principal series $\mathrm{I}(s, \Pi_{\varphi_i} \otimes \Sigma)$ reduces at $s = 1$ for every i. This is equivalent to

$$L(0, \Sigma \times \Pi_{\varphi_i}) = \infty.$$

In particular, it follows that π_i's are a permutation of the tame supercuspidal representations Π_{φ_i}'s as desired. The theorem is proved. □

We have the following consequence of Theorem 6.1:

Corollary 6.2. *Let k be a p-adic field with p odd. The parametrization of DeBacker–Reeder of generic depth zero supercuspidal representations of the split group $\mathrm{SO}_{2n+1}(k)$ coincides with the parametrization of Jiang and Soudry.*

Proof. This follows at once from Theorem 6.1 since Jiang and Soudry attach the Langlands parameter to a generic supercuspidal representation using the lift to $\mathrm{GL}_{2n}(k)$. □

We have similar results for $\mathrm{Sp}_{2n}(k)$. Let $\phi = \phi_1 \oplus \cdots \oplus \phi_d \oplus \chi^d$ be a regular tame discrete series parameter for $\mathrm{Sp}_{2n}(k)$ in general position. Recall that ϕ defines a cuspidal and generic representation Σ of $\mathrm{Sp}_{2n}(k)$ and a depth zero self-dual supercuspidal representation Π_{ϕ_i} of $\mathrm{GL}_{2n_i}(k)$ with central character χ for every $i = 1, \ldots, d$.

Theorem 6.3. *Let Σ be a depth zero generic supercuspidal representation of $\mathrm{Sp}_{2n}(k)$ corresponding, in the sense of DeBacker and Reeder, to a tame regular discrete series symplectic parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_d \oplus \chi^d$. Then*

$$\mathrm{Ind}\big(\Pi_{\phi_1} \otimes \cdots \otimes \Pi_{\phi_d} \otimes \chi^d\big)$$

is the lift Π_Σ of Σ to $\mathrm{GL}_{2n+1}(k)$, in the sense of Theorem 7.3 in [CKPS]. In particular,

$$\begin{cases} L(s, \Sigma \times \pi) = \prod_{i=1}^{d} L(s, \Pi_{\phi_i} \times \pi) \cdot L\big(\chi^d \times \pi\big), \\ \epsilon(s, \Sigma \times \pi) = \prod_{i=1}^{d} \epsilon(s, \Pi_{\phi_i} \times \pi) \cdot \epsilon\big(s, \chi^d \times \pi\big) \end{cases}$$

holds for any representation π of any $\mathrm{GL}_m(k)$.

Proof. The proof is similar to the one for $\mathrm{SO}_{2n+1}(k)$. Cogdell et al. show that there exist self-dual, mutually non-isomorphic cuspidal representations π_i of $\mathrm{GL}_{m_i}(k)$ $(m_1 + \cdots + m_r = 2n + 1)$ such that Σ lifts to a representation

$$\Pi_\Sigma = \mathrm{Ind}(\pi_1 \otimes \cdots \otimes \pi_r).$$

As in the case of $\mathrm{SO}_{2n+1}(k)$ we can show that all Π_{ϕ_i} are amongst π_i's. The remaining π_i is a one-dimensional character. Since the lift of Σ has the trivial central character, the remaining π_i must be χ^d. The theorem is proved. □

We finish by pointing out some consequences of Theorem 6.3. Let $L(s,\Sigma)$ be the standard L-function of Σ. Then, by taking $\pi = 1$ in the L-function identity in Theorem 6.3,

$$L(s,\Sigma) = \prod_{i=1}^{d} L(s,\Pi_{\phi_i}) \cdot L\big(s,\chi^d\big).$$

Since $L(s,\Pi_{\phi_i}) = 1$ (the standard L-function of a supercuspidal representation of $\mathrm{GL}_{2m}(k)$ is trivial) we obtain a formula for the standard L-function of Σ:

$$L(s,\Sigma) = \frac{1}{1-\frac{(-1)^d}{p^s}}.$$

This coincides with a result of J. Kim in [Ki]. There he calculated the standard L-function for supercuspidal representations of $\mathrm{Sp}_{2n}(k)$ in some low rank cases and $d = 1$, that is, for depth zero supercuspidal representations attached to the Coxeter torus. In particular, if d is even, then $L(0,\Sigma) = \infty$. It follows, from [MS], that Σ is a theta lift of a generic discrete series of the split group SO_{2n} with the parameter $\phi_1 \oplus \cdots \oplus \phi_d$. (The parameter here is in the sense of Cogdell, Kim, Piatetski-Shapiro and Shahidi.)

Acknowledgments

This note is motivated by and provides a building block for the approach to the inverse Galois problem initiated in [KLS]. We have benefited from the recent work of Kutzko and Morris [KM]. We would like to thank G. Henniart for providing us with a reference [He1] and the referee for careful reading of this paper and suggestions. This work is supported by NSF grant DMS 0551846.

References

[Ca] R. Carter, Finite Groups of Lie Type, Wiley Classics Lib., Wiley–Intersci., New York, 1993.
[CKPS] J. Cogdell, H. Kim, I. Piatetski-Shapiro, F. Shahidi, Functoriality for the classical groups, Publ. Math. Inst. Hautes Études Sci. 99 (2004) 163–233.
[DR] S. DeBacker, M. Reeder, Depth zero supercuspidal L-packets and their stability, Ann. of Math., in press.
[GR] B.H. Gross, M. Reeder, From Laplace to Langlands via representations of orthogonal groups, Bull. Amer. Math. Soc. 43 (2006) 163–205.
[He1] G. Henniart, Correspondence de Langlands–Kazhdan explicite dans le cas non ramifié, Math. Nachr. 158 (1992) 7–26.
[He2] G. Henniart, Correspondence de Langlands et fonctions L des carrés extérieur et symt́rique, preprint, Inst. Hautes Études Sci., 2003.
[HL] R.B. Howlett, G.I. Lehrer, Induced representations and generalized Hecke rings, Invent. Math. 58 (1980) 37–64.
[J] D. Jiang, Residues of Eisenstein Series and Related Problems, Eisenstein Series and Applications, Birkhäuser, 2007, in press.
[JS] D. Jiang, D. Soudry, The local converse theorem for $\mathrm{SO}(2n+1)$ and applications, Ann. of Math. (2) 157 (3) (2003) 743–806.
[KLS] C. Khare, M. Larsen, G. Savin, Functoriality and the inverse Galois problem, Compos. Math., in press.
[Ki] J. Kim, Gamma factors of certain supercuspidal representations, Math. Ann. 317 (4) (2000) 751–781.
[KM] P. Kutzko, L. Morris, Level zero Hecke algebras and parabolic induction: The Siegel case for split classical groups, Int. Math. Res. Not. 2006 (2006) 1–40.
[Lu] G. Lusztig, Characters of Reductive Groups over Finite Fields, Ann. of Math. Stud., vol. 107, Princeton Univ. Press, Princeton, NJ, 1984.

[Ma] H. Matsumoto, Analyse harmonique dans les systémes de Tits bornologiques de type affine, Lecture Notes in Math., vol. 590, Springer-Verlag, Heidelberg, 1977.

[Moe] C. Moeglin, Points de reducibilité par les induites de cuspidales, J. Algebra 268 (1) (2003) 81–117.

[Mor] L. Morris, Tamely ramified Hecke algebras, Invent. Math. 114 (1993) 1–54.

[Moy] A. Moy, The irreducible orthogonal and symplectic Galois representations of a p-adic field (the tame case), J. Number Theory 10 (1984) 341–344.

[Mu] G. Muić, Some results on square integrable representations: Irreducibility of standard representations, Int. Math. Res. Not. 1998 (14) (1998) 705–726.

[MS] G. Muić, G. Savin, Symplectic-orthogonal theta lifts of generic discrete series, Duke Math. J. 101 (2000) 317–333.

[Sh] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, Ann. of Math. 132 (1990) 273–330.