

FUNCTORIALITY AND THE INVERSE GALOIS PROBLEM II: GROUPS OF TYPE B_n AND G_2

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1. INTRODUCTION

1.1. **Earlier work.** Let ℓ be a prime. In our previous work [KLS], which generalised a result of Wiese [W], the Langlands functoriality principle was used to show that for every positive integer t there exists a positive integer k divisible by t such that the finite simple group $C_n(\ell^k) = \mathrm{PSp}_{2n}(\mathbb{F}_{\ell^k})$ is the Galois group of an extension of \mathbb{Q} unramified outside $\{\infty, \ell, q\}$ where $q \neq 2$ is a prime that depends on t . The construction is based on the following three steps.

- (1) Starting with a cuspidal automorphic representation on the split group SO_{2n+1} constructed using the Poincaré series, we use the global lift of Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPS] and results of Jiang and Soudry [JS1] to obtain a self-dual cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A})$, with \mathbb{A} the adèles of \mathbb{Q} , such that the following three conditions hold:
 - Π_∞ is cohomological.
 - Π_q is a supercuspidal representation of depth 0.
 - Π_v is unramified for all primes $v \neq \ell, q$.
- (2) The work of Kottwitz, Clozel, Harris-Taylor and Taylor-Yoshida yields the following theorem (see Theorem 3.6 of [Ty] or Theorem 1.1 of [Ha]). We use the conventions and notations of §1 of [Ha].

Theorem 1.1. *Let Π be a self-dual cuspidal automorphic representation Π of $\mathrm{GL}_m(\mathbb{A})$ such that Π_∞ is cohomological. Assume that for some finite place v_0 of \mathbb{Q} , Π_{v_0} is square integrable. Then attached to π and a choice of an embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$, there is an irreducible ℓ -adic representation $r'_\Pi : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_m(\bar{\mathbb{Q}}_\ell)$ of the Galois group $G_{\mathbb{Q}}$ of \mathbb{Q} such that for all primes v of \mathbb{Q} of residue characteristic $\neq \ell$ we have:*

$$WD_v(r'_\Pi)^{\mathrm{Frob-ss}} = \mathcal{L}(\Pi_v \otimes | \cdot |_{v^{\frac{1-m}{2}}})$$

Here $WD_v(r'_\Pi)$ is the Weil-Deligne parameter of $r'_\Pi|_{D_v}$ with D_v a decomposition group at v , \mathcal{L} is the normalised local Langlands correspondence, and $\mathrm{Frob-ss}$ denotes Frobenius semisimplification.

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Remark: Let χ_ℓ be the ℓ -adic cyclotomic character. If $m = 2n + 1$ we consider a twist $r_\Pi = r'_\Pi \otimes \chi_\ell^n$ and note that we have

$$WD_v(r_\Pi)^{\text{Frob-ss}} = \mathcal{L}(\Pi_v).$$

- (3) The last step consists of reducing r'_Π modulo ℓ . The parameter of Π_q can be picked so that $r_\Pi(G_{\mathbb{Q}_q})$ is a metacyclic group deeply embedded in $r'_\Pi(G_{\mathbb{Q}})$ [KW]. That is, for some large positive integer d , $r'_\Pi(G_{\mathbb{Q}_q})$ is contained in every normal subgroup of $r'_\Pi(G_{\mathbb{Q}})$ of index less than or equal to d . This property is crucial to assure, using the main result of [LP], that the reduction modulo ℓ is a simple group of type $\text{PSp}_{2n}(\mathbb{F}_{\ell^k})$.

1.2. Main theorem. The purpose of this work is to extend these results and to construct finite simple groups of type B_n and G_2 as Galois groups over \mathbb{Q} . In the case of G_2 , our result depends on a recent technical improvement of Theorem 1.1 due to Shin [Sh] in the case that m is odd. He shows that we may drop the hypothesis of the existence of a place v_0 such that Π_{v_0} is square integrable. The resulting representation r'_Π is semi-simple although it is expected to be irreducible.

We can state our main theorem:

Theorem 1.2. *Let t be a positive integer. We assume that t is even if $\ell = 3$ in the first case below:*

- (1) *Let ℓ be a prime. Then there exists an integer k divisible by t such that the simple group $G_2(\mathbb{F}_{\ell^k})$ appears as a Galois group over \mathbb{Q} .*
- (2) *Let ℓ be an odd prime. Then there exists an integer k divisible by t such that the finite simple group $\text{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\text{der}}$ or the finite classical group $\text{SO}_{2n+1}(\mathbb{F}_{\ell^k})$ appears as a Galois group over \mathbb{Q} .*
- (3) *If $\ell \equiv 3, 5 \pmod{8}$, then there exists an integer k divisible by t such that the finite simple group $\text{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\text{der}}$ appears as a Galois group over \mathbb{Q} .*

1.3. Sketch of proof. The construction of Galois groups in Theorem 1.2 is based on the functorial lift from Sp_{2n} to GL_{2n+1} [CKPS] plus the lift from G_2 to Sp_6 using the theta correspondence arising from the minimal representation of the exceptional group E_7 (see [Sa1] for a definition of the minimal representation). The main new technical difficulty in implementing the strategy of [KLS] in the present case, is that $\text{GL}_{2n+1}(\mathbb{Q}_p)$ has self-dual supercuspidal representations only if $p = 2$. Thus, while we can still construct a self-dual cuspidal automorphic representation Π of GL_{2n+1} which should give rise to our desired Galois groups, the local component Π_q cannot be supercuspidal. For groups of type B_n we can remedy the situation by requiring that the local component Π_2 be supercuspidal (which we pick to be of depth one). Existence of a global Π with such local component Π_2 is again obtained using the global lift from Sp_{2n} plus recently announced backward lift from GL_{2n+1} to Sp_{2n} by Jiang and Soudry [JS2]. The local component Π_2 not only assures us of the existence of the ℓ -adic representation r_Π , without using new results of Shin, but it also gives us a certain control over the Galois group obtained by reducing r_Π modulo ℓ . More precisely, Π_2 can be picked so that the image of the local Langlands parameter is a finite group I in $\text{GL}_{2n+1}(\mathbb{C})$ with the following properties:

- $I/[I, I] \cong \mathbb{Z}/(2n + 1)\mathbb{Z}$.
- $[I, I] \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$.

If $\ell \equiv 3, 5 \pmod{8}$ then the first property of I implies that the Galois group is $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ and not $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$. If $n = 3$ then the second property of I implies that Π_2 is not a lift from $\mathrm{G}_2(\mathbb{Q}_2)$ and the Galois group is not $\mathrm{G}_2(\ell^k)$.

Correction to [KLS] : With the definition of the group of type (n, p) in [KLS], to ensure that $\bar{\rho}(D_q)$ in §5.2 be of type (n, p) we should ask that the K of §3.3 also contain $\mathbb{Q}(\zeta_\ell)$, in addition to the other conditions there. Alternatively, and better, the definition of a group of type (n, p) in [KLS] could be modified (and made less restrictive) as in Definition 10.2 below. All statements in [KLS] then go through with this altered definition, with obvious modifications in their proof.

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2. LOCAL DISCRETE SERIES PARAMETERS

Let k be a local field and G a connected reductive and split group over k . Conjecturally, representations of $G(k)$ correspond to (certain) homomorphisms of the Weil-Deligne group

$$\phi : WD_k \rightarrow G^*(\mathbb{C})$$

into the Langlands dual group $G^*(\mathbb{C})$. In this paper we shall be concerned with the following cases:

G	GL_n	Sp_{2n}	PGSp_6	G_2
G^*	GL_n	SO_{2n+1}	Spin_7	G_2

If $G = \mathrm{Sp}_{2n}$, it will be convenient to realize the dual group as $\mathrm{SO}(U)$ for some choice of a non-degenerate complex orthogonal space U of dimension $2n + 1$. Then a discrete series parameter for $\mathrm{Sp}_{2n}(k)$ is a homomorphism $\phi : WD_k \rightarrow \mathrm{SO}(U)$ such that under the action of WD_k the orthogonal space U decomposes into irreducible summands

$$U = U_1 \oplus \cdots \oplus U_s$$

where each U_i is a non-degenerate orthogonal subspace of U . Moreover, if ϕ_i denotes the representation of WD_k on U_i , then $\phi_i \cong \phi_j$ if and only if $i = j$. In other words, we are requiring that the image of WD_k is not contained in a proper Levi factor in G^* .

Consider now the case $k = \mathbb{R}$. In this case the Weil-Deligne group is the same as the Weil group $W_{\mathbb{R}}$. For every non-zero integer a let η_a be a character of $W_{\mathbb{C}} \cong \mathbb{C}^\times$ defined by

$$\eta_a(z) = \left(\frac{z}{\bar{z}}\right)^a.$$

Let

$$\phi(a) = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \eta_a.$$

This is an irreducible and orthogonal 2 dimensional representation of $W_{\mathbb{R}}$. Its determinant is the unique non-trivial quadratic character χ_∞ of $W_{\mathbb{R}}^{ab} \cong \mathbb{R}^\times$. If a_1, \dots, a_n are non-zero integers such that $a_i \neq \pm a_j$, then

$$\phi(a_1, \dots, a_n) \oplus \chi_\infty^n.$$

is a discrete series parameter for the group $\mathrm{Sp}_{2n}(\mathbb{R})$. Note that the choice of exponent $-n$ in the last summand is made so that the image of the parameter is contained in $\mathrm{SO}_{2n+1}(\mathbb{C})$. Note that the parameter is determined by a_i up to permutation of indices and change of signs. If $n = 3$, then the image of the parameter is contained in $\mathrm{G}_2(\mathbb{C}) \subset \mathrm{SO}_7(\mathbb{C})$ if and only if

$$a_1 + a_2 + a_3 = 0$$

for some choices of signs of a_i 's. Let σ_∞ be a generic discrete series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ (or of $\mathrm{G}_2(\mathbb{R})$) corresponding to this parameter.

Let Π_∞ be the lift of σ_∞ to $\mathrm{GL}_{2n+1}(\mathbb{R})$. The infinitesimal character of Π_∞ is represented by a $2n + 1$ -tuple

$$(a_1, \dots, a_n, -a_1, \dots, -a_n, 0).$$

In particular, Π_∞ is cohomological, as defined by Clozel [Cl].

3. DEPTH ZERO GENERIC SUPERCUSPIDAL REPRESENTATIONS

Let q be an odd prime. Let $\Omega_{q'}$ denote the set of all complex roots of unity of order prime to q . The Frobenius acts on $\Omega_{q'}$ by

$$F(\tau) = \tau^q$$

for every τ in $\Omega_{q'}$. Note that all F -orbits are finite. These orbits play a key role in the description of tame parameters.

Lemma 3.1. *Let τ be a root of 1 different from ± 1 . Assume that the F -orbit of τ has m different elements:*

$$\tau, \tau^q, \dots, \tau^{q^{m-1}}.$$

If τ^{-1} is on this list, that is, if $\tau^{-1} = \tau^{q^n}$ for some $n < m$ then $m = 2n$.

Proof. First of all, note that $0 < n$ since $\tau \neq \pm 1$. Raising $\tau^{-1} = \tau^{q^n}$ to the q^n -th power gives $\tau = \tau^{q^{2n}}$. Since $\tau = \tau^{q^k}$ if and only if k is a multiple of m , and $0 < 2n < 2m$, it follows that $m = 2n$, as claimed. \square

We are now ready to define irreducible tame self-dual parameters of $\mathrm{Sp}_{2n}(\mathbb{Q}_q)$. Let $\mathbb{Q}_{q^{2n}}$ be the unique unramified extension of \mathbb{Q}_q of degree $2n$. Then

$$\mathbb{Q}_{q^{2n}}^\times = \langle q \rangle \times \mathbb{F}_{q^{2n}}^\times \times U_1$$

where U_1 is the maximal pro q -subgroup of $\mathbb{Q}_{q^{2n}}^\times$. A character of $\mathbb{Q}_{q^{2n}}^\times$ is called tame if it is trivial on U_1 . Let ζ_{2n} be a primitive root in $\mathbb{F}_{q^{2n}}^\times$. Pick τ , a complex root of 1 such that the F -orbit τ, τ^q, \dots has precisely $2n$ distinct elements and $\tau^{q^n} = \tau^{-1}$. (For example, τ can be picked a primitive root of order $q^n + 1$.) Then τ defines a tame character η of $\mathbb{Q}_{q^{2n}}^\times$ by

$$\begin{cases} \eta(\zeta_{2n}) = \tau \\ \eta(q) = 1. \end{cases}$$

Let $W_{\mathbb{Q}_q}$ and $W_{\mathbb{Q}_{q^{2n}}}$ be the local Weil groups of \mathbb{Q}_q and $\mathbb{Q}_{q^{2n}}$. Recall that

$$W_{\mathbb{Q}_q}/W_{\mathbb{Q}_{q^{2n}}} \cong \mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_q).$$

Via the local class field theory we have an identification $W_{\mathbb{Q}_q^{2n}}^{ab} \cong \mathbb{Q}_q^{\times}$. Note that $\eta \circ F^i \neq \eta$ for $1 < i \leq 2n$ and $\eta \circ F^n = \bar{\eta}$. In particular, the character η defines an irreducible, orthogonal $2n$ -dimensional representation

$$\phi(\tau) = \text{Ind}_{W_{\mathbb{Q}_q^{2n}}}^{W_{\mathbb{Q}_q}}(\eta).$$

of $W_{\mathbb{Q}_q}$. We note that the determinant of $\phi(\tau)$ is the unique unramified quadratic character χ_q of $W_{\mathbb{Q}_q}^{ab} \cong \mathbb{Q}_q^{\times}$.

Pick a sequence τ_1, \dots, τ_s of roots in Ω_{q^i} belonging to different F -orbits of order $2n_1, \dots, 2n_s$ such that $\tau^{q^{n_i}+1} = 1$ for every i and $2n_1 + \dots + 2n_s = 2n$. Corresponding to this we have a tame regular discrete series parameter for the split group $\text{Sp}_{2n}(\mathbb{Q}_q)$

$$\phi = \phi(\tau_1, \dots, \tau_s) \oplus \chi_q^s$$

where, as in the case of real groups, the exponent d is picked to assure that the image of the parameter is contained in $\text{SO}_{2n+1}(\mathbb{C})$. Note that the image $\phi(I_q)$ of the inertia subgroup $I_q \subseteq W_{\mathbb{Q}_q}$ is contained in a maximal torus of $\text{SO}_{2n+1}(\mathbb{C})$ and $\phi(F)$ is an elliptic element of the Weyl group. If $s = 1$, for example, then the image of the inertia is a cyclic group generated by an element whose eigenvalues are

$$\tau, \tau^q, \dots, \tau^{q^n}, \tau^{-1}, \dots, \tau^{-q^n}, 1$$

and $\phi(F)$ correspond to the Coxeter element in the Weyl group.

Proposition 3.2. *The image of a tame regular discrete series parameter $\phi = \phi(\tau_1, \dots, \tau_s) \oplus \chi_q^s$ of $\text{Sp}_6(\mathbb{Q}_q)$ is contained in $\text{G}_2(\mathbb{C})$ if and only if one of the two holds:*

- (1) $s = 3$ and $\tau_1\tau_2\tau_3 = 1$, for some choices of τ_i^{\pm} (F -orbit of τ_i consists of τ_i and τ_i^{-1}).
- (2) $s = 1$ and τ satisfies $\tau^{q^2-q+1} = 1$. (Recall that τ , a priori, satisfies a weaker condition $\tau^{q^3+1} = 1$.)

Proof. The weights of the 7-dimensional representation of $\text{G}_2(\mathbb{C})$ are 0 and six short roots. Pick three short roots α_1, α_2 and α_3 such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. If t is a semi-simple element in $\text{G}_2(\mathbb{C})$, put $\lambda_i^{\pm} = \pm\alpha_i(t)$. Then $\lambda_1^{\pm}, \lambda_2^{\pm}, \lambda_3^{\pm}$ and 1 are the eigenvalues of t in the 7-dimensional representation. Note that $\lambda_1\lambda_2\lambda_3 = 1$.

If the parameter ϕ is contained in $\text{G}_2(\mathbb{C})$ then $\phi(F)$ corresponds to a Weyl group element in G_2 of even order. Since 2 and 6 are only even orders of elements in the Weyl group of G_2 , we see that $s = 1$ or 3. If $s = 3$ then $\phi(F)$ corresponds to -1 in the Weyl group and the condition $\lambda_1\lambda_2\lambda_3 = 1$ translates into $\tau_1\tau_2\tau_3 = 1$. If $s = 1$ then $\phi(F)$ corresponds to the Coxeter element. We can pick the Coxeter element (or alternatively the roots α_i) so that it cyclically permutes the roots

$$\alpha_1, -\alpha_2, \alpha_3, -\alpha_1, \alpha_2, -\alpha_3.$$

On the other hand, $\phi(I_q)$ is generated by a semi-simple element t with non-trivial eigenvalues $\tau, \tau^q, \tau^{q^2}, \tau^{-1}, \tau^{-q}, \tau^{-q^2}$ which $\phi(F)$ permutes cyclically in the given order. In particular, the condition $\lambda_1\lambda_2\lambda_3 = 1$ translates into $\tau^{1-q+q^2} = 1$, as desired. Conversely, if the parameter satisfies the conditions of (1) and (2) then we can factor ϕ through $\text{G}_2(\mathbb{C})$ since -1 and the Coxeter element can be lifted from the Weyl group to $\text{G}_2(\mathbb{C})$. \square

Let σ_q be a generic supercuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{Q}_q)$ (or of $\mathrm{G}_2(\mathbb{Q}_q)$) corresponding, via DeBacker-Reeder, to a tame parameter as above. Then the lift of σ_q to $\mathrm{GL}_{2n+1}(\mathbb{Q}_q)$ [Sa2] is

$$\Pi_1 \times \cdots \times \Pi_s \times \chi_q^s.$$

This is a tempered representation parabolically induced from supercuspidal representations Π_1, \dots, Π_s corresponding to irreducible tame parameters $\phi(\tau_1), \dots, \phi(\tau_s)$ by the local Langlands correspondence [HT]. We note that the recipe of DeBacker-Reeder [DR] involves picking a hyperspecial compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$. Since there are two non-conjugate hyperspecial maximal compact subgroups here, there are two possible σ_q . They have the same lift to $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$.

Of interest to us is the parameter of type $\phi(\tau) \oplus \chi_q$ where q and τ are picked using the following lemma (Lemma 3.4 in [KLS]):

Lemma 3.3. *Given a positive integer $m = 2n$, a prime ℓ , a finite Galois extension K of \mathbb{Q} , and positive integers t and d , there exists odd primes p and q such that*

- (1) *Primes ℓ , p and q are all distinct.*
- (2) *The prime p is greater than d .*
- (3) *If $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$ contains an element of order p then \mathbb{F}_{ℓ^k} contains \mathbb{F}_{ℓ^t} . In particular, t divides k .*
- (4) *The prime q splits completely in K .*
- (5) *The order of q in \mathbb{F}_p^\times is exactly m .*

We remark that if $n = 3$, $\ell = 3$, and t is even, then no Ree group ${}^2\mathrm{G}_2(\mathbb{F}_{3^{2f+1}})$ contains an element of order p . Indeed,

$${}^2\mathrm{G}_2(\mathbb{F}_{3^{2f+1}}) < \mathrm{G}_2(\mathbb{F}_{3^{2f+1}}) < \mathrm{SO}_7(\mathbb{F}_{3^{2f+1}}),$$

and t does not divide $2f + 1$.

Let K be the composite of all Galois extensions of \mathbb{Q} of degree $\leq d$ and ramified at $2, \ell$ and no other primes. This is a finite degree Galois extension of \mathbb{Q} ramified at $2, \ell$ and no other primes. Let p and q be the primes given by Lemma 3.3 applied to this field K . Let τ be a primitive p -th root of 1. Since the order of q in \mathbb{F}_p^\times is precisely $2n$, the Fr_q -orbit of τ gives rise to a tame parameter $\phi(\tau) \oplus \chi_q$. Moreover, if $n = 3$ then $\tau^{q^2 - q + 1} = 1$ since τ is of order p and p , by construction, divides $\Phi_6(q) = q^2 - q + 1$. In particular, this parameter is automatically a G_2 -parameter. In any case, we note that the image of the inertia I_q subgroup is a cyclic group of order p . The image of the Weil group is a semi-direct product of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ and the cyclic group $\mathbb{Z}/2n\mathbb{Z}$. This group is also called a *metacyclic* group and denoted by $\Gamma_{2n,p}$.

4. IRREDUCIBLE SUPERCUSPIDAL PARAMETERS

As we have seen in the previous section, the image of a tame supercuspidal parameter $\varphi : W_k \rightarrow G^*$ is not irreducible when acting on the standard representation U of G^* . In particular, the lift to $\mathrm{GL}_n(k)$ ($n = \dim(U)$) of the corresponding supercuspidal representation is not supercuspidal. In order to remedy this, we need to introduce certain wildly ramified parameters. This will be done using (so-called) Jordan subgroups of the complex reductive group G^* . A Jordan subgroup J of G^* is an elementary abelian p -subgroup such that its

normalizer N in G^* is a finite subgroup and J is a minimal normal subgroup of N see [KT], page 505. The following is a partial list of Jordan subgroups.

G^*	J	N/J
SO_{2n+1}	$(\mathbb{F}_2)^{2n}$	S_{2n+1}
G_2	$(\mathbb{F}_2)^3$	$\mathrm{SL}_3(2)$

Here S_{2n+1} is the symmetric group of $2n + 1$ letters. We note that the conjugation action of N/J on J given by the standard representation of N/J on J . (In the first case we mean by this the restriction of the permutation representation of S_{2n+1} on \mathbb{F}_2^{2n+1} to the hyperplane given by $\sum_{i=1}^{2n+1} x_i = 0$.) However the extension of N/J by J is not necessarily split.

We shall now construct a map $\varphi : W_{\mathbb{Q}_p} \rightarrow G^*$ such that the image of the wild inertia is J (in particular $p = 2$) and the image of $W_{\mathbb{Q}_p}$ is an intermediate subgroup $J \subseteq I \subseteq N$ acting irreducibly on the standard representation of G^* .

Let us consider the case $G^* = \mathrm{SO}_{2n+1}(\mathbb{C})$ first. Let us abbreviate $m = 2n + 1$, and let \mathbb{Q}_{2^m} be the unramified extension of \mathbb{Q}_2 of degree m . Then

$$\mathbb{Q}_{2^m}^\times = \langle 2 \rangle \times \mathbb{F}_{2^m}^\times \times U$$

where U is a pro-2 group with a filtration $U \supset U_1 \supset U_2 \dots$ such that $U/U_1 \cong \mathbb{F}_{2^m}$. Let e be a primitive element in \mathbb{F}_{2^m} . Let $e_i = \mathrm{Fr}_2^{i-1}(e)$. Then $e = e_1, e_2, \dots, e_m$ give a basis of \mathbb{F}_{2^m} over \mathbb{F}_2 . In particular, we have fixed an isomorphism

$$U/U_1 \cong (\mathbb{F}_2)^m.$$

In this way any character of U/U_1 can be viewed as an m -tuple of signs. Let χ be the character corresponding to the m -tuple $(-, -, +, \dots, +)$. We extend χ to $\mathbb{Q}_{2^m}^\times$ so that it is trivial on the first two factors. Since $W_{\mathbb{Q}_{2^m}}^{ab} \cong \mathbb{Q}_{2^m}^\times$ we can view χ as a character of $W(K)$. Define

$$\phi_2 = \mathrm{Ind}_{W_{\mathbb{Q}_{2^m}}}^{W_{\mathbb{Q}_2}}(\chi).$$

Since the conjugates $\chi \circ \mathrm{Fr}_2^i$, for $i = 1, \dots, m$, are mutually distinct this representation is irreducible by Mackey's criterion. Since χ is quadratic the representation ϕ_2 is also self-dual and, since m is odd, it is orthogonal. Thus ϕ_2 defines a self-dual supercuspidal representation Π_2 of $\mathrm{GL}_{2n+1}(\mathbb{Q}_2)$ by the local Langlands correspondence.

For later purposes we need to describe the image of the representation ϕ_2 . Note that the intersection of the kernels of $\chi \circ \mathrm{Fr}_2^i$ is equal to $\Delta\mathbb{F}_2$, the diagonal in \mathbb{F}_2^m .

Proposition 4.1. *Recall that $m = 2n + 1$. Let I be the image of $W(\mathbb{Q}_2)$ under the representation ϕ_2 . Then*

- (1) $I/[I, I] \cong \mathbb{Z}/m\mathbb{Z}$.
- (2) $[I, I] \cong \mathbb{F}_2^m/\Delta(\mathbb{F}_2)$.
- (3) I is contained in a special orthogonal group.
- (4) If $m = 7$ then I is not contained in G_2 .

Proof. Since $W_{\mathbb{Q}_2}/W_{\mathbb{Q}_{2^m}}$ is a cyclic group of order m , in order to prove the first two statements, it suffices to show that the commutator is given by the image of $W_{\mathbb{Q}_{2^m}}$. Note that the commutator of e_i and Fr_2 in $W_{\mathbb{Q}_2}$ is equal to $e_i + e_{i+1}$, considered as an element of $U/U_1 \cong \mathbb{F}_2^m$. Since m is odd, these elements generate $\mathbb{F}_2^m/\Delta(\mathbb{F}_2)$. The first two statements

now follow. Since the determinant character is of order two and $I/[I, I]$ is odd, it has to be trivial on I . This shows the third statement. Finally, if $m = 7$, then $\phi_2(e_i)$ has eigenvalues 1 (with multiplicity 5) and -1 (with multiplicity 2). These cannot be arranged as a family of type $\lambda_1^{\pm 1}, \lambda_2^{\pm}, \lambda_3^{\pm}$ and 1 such that $\lambda_1 \lambda_2 \lambda_3 = 1$. The proposition is proved. \square

We now consider the Jordan subgroup in G_2 . the intermediate group $J \subseteq I \subseteq N$ in advance so that I/J is the normalizer of an elliptic torus in $N/J \cong \mathrm{SL}_3(2)$. In particular, if we identify J with \mathbb{F}_{2^3} then I/J can be identified as a semi-direct product of $\mathrm{Gal}(\mathbb{F}_{2^3}/\mathbb{F}_2)$ and $\mathbb{F}_{2^3}^\times$. Since the order of I/J is prime to J one easily check that this extension splits. Note that I/J acts transitively on the set of non-trivial irreducible characters of J . Let ψ be a non-trivial additive character of \mathbb{F}_2 . Then the composition of the trace $Tr : \mathbb{F}_{2^3} \rightarrow \mathbb{F}_2$ is a character of \mathbb{F}_{2^3} such that its stabilizer in I/J is $\mathrm{Gal}(\mathbb{F}_{2^3}/\mathbb{F}_2)$. It follows, from Mackey's theory, that I has 3 irreducible faithful representations, of dimension 7, only one of which is self-dual.

Proposition 4.2. *Let $\varphi : W_{\mathbb{Q}_2} \rightarrow G_2$ be a parameter with the image I . Let $\phi_2 : W_{\mathbb{Q}_2} \rightarrow \mathrm{GL}_7(\mathbb{C})$ obtained by natural inclusion $G_2 \subseteq \mathrm{GL}_7(\mathbb{C})$. Then ϕ_2 is a self-dual, irreducible representation of $W_{\mathbb{Q}_2}$.*

Proof. This is easy. We know that any irreducible representation of I either has J in the kernel or it is faithful, in which case it is of the dimension $2^3 - 1 = 7$. There are three such representations, only one is self-dual. Since the restriction of the standard representation of G_2 to I is faithful and self-dual, it must be isomorphic to the unique irreducible self-dual representation of I of the same dimension. \square

It remains to show that the group I can be obtained as the image of the Weil group $W_{\mathbb{Q}_2}$. Let L be the Galois extension of \mathbb{Q}_2 given as the totally ramified extension of \mathbb{Q}_{2^3} of degree 7. In other words, L is the splitting field of the polynomial

$$X^7 - 2 = 0.$$

Note that the Galois group of L is isomorphic to I/J . Let ϖ be a uniformizer in L , $U \subseteq L^\times$ the maximal pro-2 subgroup and $U \supseteq U_1 \supseteq \dots$ the usual filtration. Then

$$L^\times = \langle \varpi \rangle \times \mathbb{F}_{2^3}^\times \times U.$$

Let χ be a character of $W_L^{ab} \cong L^\times$ which is trivial on the the first two factors of L^\times and a non-trivial character of $U/U_1 \cong \mathbb{F}_{2^3}$. Consider the induced representation

$$\mathrm{Ind}_{W_L}^{W_{\mathbb{Q}_2}}(\chi).$$

This representation breaks up as a sum of three irreducible representations of dimension 7, one of which is self-dual. W_K be the kernel of this representation. Then the Galois group of K over \mathbb{Q}_2 is isomorphic to I . In other words, we have constructed map $\varphi : W_{\mathbb{Q}_2} \rightarrow G^*$ with the image I .

5. LOCAL LIFT FROM G_2 TO PGSp_6

The dual group of $\mathrm{PGSp}_6(\mathbb{Q}_p)$ is $\mathrm{Spin}_7(\mathbb{C})$. The group $\mathrm{Spin}_7(\mathbb{C})$ has a unique open orbit on the 8-dimensional spin representation. The stabilizer of a point in the open orbit is isomorphic to $G_2(\mathbb{C})$. This gives an embedding

$$f : G_2(\mathbb{C}) \rightarrow \mathrm{Spin}_7(\mathbb{C})$$

of dual groups, indicating that there should be a functorial, but non-endoscopic, lift of representations from $G_2(\mathbb{Q}_p)$ to $\mathrm{PGSp}_6(\mathbb{Q}_p)$, once local Langlands parametrizations for the two groups are established. The Langlands parameterization is essentially known for depth zero representations. We shall now spell out some special cases of our interest. In order to simplify notation let

$$\begin{cases} G = G_2(\mathbb{Q}_p) \\ G' = \mathrm{PGSp}_6(\mathbb{Q}_p). \end{cases}$$

Recall that a root of one τ such that $\tau^{p^2-p+1} = 1$ defines a 7-dimensional orthogonal parameter $\phi(\tau) \oplus \chi_p$ which is contained in $G_2(\mathbb{C})$. Therefore, it defines a generic supercuspidal representation denoted by $\sigma(\tau)$ of G and, by composing this parameter with the inclusion f , a generic supercuspidal representation $\sigma'(\tau)$ of G' . The representation $\sigma'(\tau)$, when restricted to $\mathrm{Sp}_6(\mathbb{Q}_p)$, breaks up as a sum of two representations in the L -packet for the parameter $\phi(\tau) \oplus \chi_p$.

We have the following:

- The functorial lift of the supercuspidal representation $\sigma(\tau)$ is the supercuspidal representation $\sigma'(\tau)$.
- The functorial lift of the Steinberg representation st_G is the Steinberg representation $\mathrm{st}_{G'}$.
- Let σ be an unramified representation σ of G corresponding to a semi-simple conjugacy class (Satake parameter) $s \in G_2(\mathbb{C})$. Then the lift of σ is σ' , an unramified representation of G' corresponding to the parameter $s' = f(s)$.

Although the local parametrizations for G and G' are not complete, a lift from G to G' is given by a correspondence arising from the minimal representation Σ of the split, adjoint $E_7(\mathbb{Q}_p)$. More precisely, if σ is an irreducible representation of G , then we define $\Theta(\sigma)$ to be the set of isomorphism classes of all irreducible representations σ' of G' such that $\sigma \otimes \sigma'$ is a quotient of Σ .

Let $\psi : U \rightarrow \mathbb{C}^\times$ be a Whittaker character for G , where U is a maximal unipotent subgroup of G . Recall that a representation σ of G is ψ -generic (or simply generic) if $\sigma_{U,\psi}$, the space of ψ -twisted U -coinvariants, is nonzero. We have the same definition for representations of G' with respect to a Whittaker character $\psi' : U' \rightarrow \mathbb{C}^\times$, where U' is a maximal unipotent subgroup of G' . Let $\Theta_{\mathrm{gen}}(\sigma)$ be the subset of $\Theta(\sigma)$ consisting of generic representations. This set is somewhat easier to determine.

Proposition 5.1. *Let σ be an irreducible representation of G . Then $\Theta_{\mathrm{gen}}(\sigma) \neq 0$ only if σ is generic. Moreover, $\Theta_{\mathrm{gen}}(\sigma)$ contains at most one element.*

Proof. Let σ' be in $\Theta_{\text{gen}}(\sigma)$. Then $\sigma \otimes \sigma'$ is a quotient of Σ . Since $\sigma'_{U', \psi'}$ is one dimensional, we see that σ is a quotient of $\Sigma_{U', \psi'}$. Since ([Ga, Theorem 7.1])

$$\Sigma_{U', \psi'} = \text{ind}_U^G(\psi),$$

as a G -module, it follows that σ is indeed generic. Moreover, since

$$\text{Hom}_G(\text{ind}_U^G(\psi), \sigma)$$

is one-dimensional for any generic representation σ (by uniqueness of the Whittaker functional), the second part follows immediately. The proposition is proved. \square

Given a generic representation σ , the above proposition allows us to show that $\Theta_{\text{gen}}(\sigma) = \{\sigma'\}$ by simply showing that $\sigma \otimes \sigma'$ is a quotient of Σ .

Proposition 5.2. *Assume that σ is an irreducible representation of $G = \text{G}_2(\mathbb{Q}_p)$, belonging to either of the following two families:*

- (1) *Supercuspidal representations $\sigma(\tau)$.*
- (2) *Generic unramified representations.*

Then $\Theta_{\text{gen}}(\sigma) \neq \emptyset$ and the unique representation in $\Theta_{\text{gen}}(\sigma)$ is the functorial lift of σ , as described above.

Proof. The supercuspidal representation $\sigma(\tau)$ is induced from a cuspidal representation ρ of the finite group $\text{G}_2(\mathbb{F}_p)$, inflated to a hyperspecial maximal compact subgroup of G . Similarly, $\sigma'(\tau)$ is induced from a cuspidal representation ρ' of $\text{PGSp}_6(\mathbb{F}_p)$. Gan shows in [Ga] that $\rho \otimes \rho'$ is a summand of the minimal representation of the adjoint group $\text{E}_7(\mathbb{F}_p)$. By [Sa1] the minimal representation of $\text{E}_7(\mathbb{F}_p)$ appears as the first non-trivial K -type of the minimal representation of $\text{E}_7(\mathbb{Q}_p)$. (Here K is a hyperspecial maximal compact subgroup in $\text{E}_7(\mathbb{Q}_p)$. In particular, $\text{E}_7(\mathbb{F}_p)$ is a quotient of K by the first congruence subgroup, and the K -type is obtained by inflating the minimal representation of $\text{E}_7(\mathbb{F}_p)$ to K .) It follows, by Frobenius reciprocity, that $\sigma(\tau) \otimes \sigma'(\tau)$ is a summand of Σ . This shows that $\Theta_{\text{gen}}(\sigma(\tau)) = \{\sigma'(\tau)\}$ as desired.

Finally, assume that σ is an unramified representation. Let $B = TU$ be a Borel subgroup of G . Then σ is a subquotient of $\text{Ind}_B^G(\chi)$ for some unramified character χ of T . (The induction is normalized here.) Recall that any root α defines a co-root homomorphism $t \mapsto h_\alpha(t)$ from \mathbb{Q}_p^\times into T . Let $\alpha_1, \alpha_2, \alpha_3$ be three short roots for G such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0.$$

This choice is unique up to the action of the Weyl group of G_2 . We can now compose χ with the co-root homomorphisms for α_i . In this way we get 3 characters χ_1, χ_2, χ_3 of \mathbb{Q}_p^\times such that $\chi_1\chi_2\chi_3 = 1$. Now, if σ is generic then (and only then) the whole induced representation is irreducible. According to a result of Muić ([Mu] Proposition 3.1) this happens if and only if

$$\chi_i \neq |\cdot|^{\pm 1} \text{ and } \chi_i/\chi_j \neq |\cdot|^{\pm 1}, 1 \leq i < j \leq 3.$$

Next, consider the representation $\pi = \chi_1 \times \chi_2 \times \chi_3$ of $\text{GL}_3(\mathbb{Q}_p)$ (here we use the notation of Bernstein and Zelevinski). Let $P = MN$ be a maximal parabolic of G' such that $M \cong \text{GL}_3(\mathbb{Q}_p)$ (see [MaS]). Then the local lift of σ is the unique unramified quotient σ' of the representation of G' obtained by inducing π . By a result of Tadić ([Ta] Theorem 7.1) this induced representation is irreducible if and only if the same conditions as those of Muić are

satisfied. In other words, an unramified representation σ is generic if and only if its local lift σ' is, and both are equal to a fully induced principal series representation. In particular, $\sigma' = \text{Ind}_P^{G'}(\pi)$ (normalized induction). By Frobenius reciprocity, we have

$$\text{Hom}_{G \times G'}(\Sigma, \sigma \otimes \sigma') = \text{Hom}_{G \times M}(\Sigma_N, \sigma \otimes \pi)$$

where Σ_N is the (normalized) Jacquet functor. Next, we recall that the minimal representation of $E_6(\mathbb{Q}_p)$ is a quotient of Σ_N [MaS] and that $\sigma \otimes \pi$ is a quotient of the minimal representation of $E_6(\mathbb{Q}_p)$ [GaS2]. It follows that $\text{Hom}_{G \times M}(\Sigma_N, \sigma \otimes \pi) \neq 0$ and $\sigma \otimes \sigma'$ is a quotient of Σ , as desired. \square

6. GLOBAL FORMS

Recall that G is the split Sp_{2n} or G_2 over \mathbb{Q} . Fix a prime ℓ and q an odd prime different from ℓ . By Theorem 4.5 [KLS] there exists a globally generic cuspidal automorphic representation σ of $G(\mathbb{A})$ such that

- σ_∞ is a generic integrable discrete series representation.
- σ_q is a tame supercuspidal generic representation; $\sigma_q = \sigma(\tau)$ if $G = G_2$.
- σ_v is unramified for all $v \neq 2, q, \ell$.
- If $G = G_2$ then σ_2 is unramified, and if $G = \text{Sp}_{2n}$, σ_2 is the Jiang-Soudry descent of the self-dual supercuspidal representation of Π_2 introduced in Section 4.

The form σ lifts to an irreducible, self-dual, automorphic representation $\text{GL}_{2n+1}(\mathbb{A})$ or $\text{GL}_7(\mathbb{A})$, with *trivial* central character. This uses the lift of Cogdell et al if G is Sp_{2n} . If G is G_2 we first use the exceptional theta lift [GRS] to obtain a non-zero generic automorphic form σ' on $\text{PGSp}_6(\mathbb{A})$. The form σ' is cuspidal if the lift of σ to $\text{PGL}_3(\mathbb{A})$ (via the minimal representation of E_6) is 0. This holds since σ_∞ is a discrete series representation and it cannot appear as a local component in the lift from $\text{PGL}_3(\mathbb{A})$ [GaS1]. Thus, σ' is a generic cuspidal automorphic representation and its local p -adic components are determined by Proposition 5.2. The infinitesimal character of the real component σ'_∞ is integral and regular by the matching of infinitesimal characters in [HPS]. Next, we restrict σ' to $\text{Sp}_6(\mathbb{A})$ and use the lift of Cogdell et al to obtain an automorphic representation Π of $\text{GL}_7(\mathbb{A})$.

Recall that χ_q is the unique non-trivial quadratic unramified character of the local Weil group. The local components of Π satisfy:

- Π_∞ has a regular and integral infinitesimal character.
- Π_q has the parameter $\phi(\tau) \oplus \chi_q$.
- Π_v is unramified for all $v \neq 2, q, \ell$.
- If $\ell \neq 2$ then Π_2 is unramified or, if $G = \text{Sp}_{2n}$, it has the irreducible parameter ϕ_2 .

Note that if Π_v is unramified then the eigenvalues of its Satake parameter are

$$\lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}, 1.$$

Moreover, if G is G_2 then we have one additional relation:

$$\lambda_1 \lambda_2 \lambda_3 = 1.$$

If Π_2 is the self-dual supercuspidal representation of $\text{GL}_{2n+1}(\mathbb{Q}_2)$ with the parameter ϕ_2 then the lift Π is clearly supercuspidal. If Π_2 is unramified then, since the parameter of σ_q is not irreducible, the global representation Π might not be cuspidal. We give a criterion which

guarantees that it is. Note that the conditions of the following proposition are automatically satisfied if q and the parameter $\phi(\tau) \oplus \chi_q$ are picked using Lemma 3.3.

Proposition 6.1. *Assume that σ is a globally generic cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$, such that σ_v is unramified at all (finite) primes $v \neq \ell, q$ and σ_q corresponds to the parameter $\phi(\tau) \oplus \chi_q$. If q splits in all quadratic extensions of \mathbb{Q} ramified at ℓ and no other primes then Π , the global lift of σ to $\mathrm{GL}_{2n+1}(\mathbb{A})$, is cuspidal.*

Proof. If Π is not cuspidal then by [CKPS] and forced by the local parameter of Π_q we have an isobaric sum

$$\Pi = \Sigma \boxplus \chi$$

where Σ is a cuspidal self-dual automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ and χ is a quadratic character of $\mathrm{GL}_1(\mathbb{A})$. The local component χ_v of χ is clearly unramified for all primes $v \neq \ell, q$. It is also unramified and non-trivial at q since it corresponds, via the local class field theory, to the one-dimensional summand of the parameter of Π_q (denoted by the same symbol χ_q). By the global class field theory χ corresponds to a quadratic extension of \mathbb{Q} ramified at ℓ and no other primes, and such that q is inert. This is a contradiction. \square

7. REDUCTIVE GROUPS

Let G be a connected reductive group over an algebraically closed field of characteristic different from 2, and let T be a maximal torus in G . A representation V of G is called almost miniscule if $V^T \neq 0$ and the Weyl group acts transitively on the set of non-trivial weights of V . These representations can be easily classified for an almost simple G . Assume first that the field characteristic is 0. Since $V^T \neq 0$ then, by Lie algebra action, V contains a root as a weight. All non-zero weights are now Weyl-group conjugates of that root. Therefore, if G is simply laced then V is the adjoint representation of G . If G is multiply laced then the weights are all short roots. We tabulate possible cases:

G	$\dim(V)$	$\dim(V^T)$
B_n	$2n + 1$	1
C_n	$2n^2 - n - 1$	$n - 1$
G_2	7	1
F_4	26	2

It is interesting to note that dimension of V^T is equal to the number of short simple roots. The Weyl group acts, naturally, on V^T . The action of long root reflections is trivial, while V^T is a reflection representation for the subgroup generated by simple short root reflections.

Proposition 7.1. *Let G be an almost simple group over an algebraically closed field of characteristic $\ell > 2$. Then:*

- (1) *Any miniscule representation is isomorphic to a Frobenius twist of a representation with a miniscule weight as the highest weight.*
- (2) *Assume first that $\ell \neq 3$ if $G = G_2$. Then any almost miniscule representation is a Frobenius twist of the almost miniscule representation with the highest weight equal to a short root. If $\ell = 3$ and $G = G_2$ then there is one additional family of miniscule representations. It consists of Frobenius twists of the representation with the highest weight equal to a long root.*

Proof. Let $\alpha_1, \dots, \alpha_r$ be the simple roots for G . We shall characterize miniscule (and then almost miniscule) representations with the highest weight λ such that $0 \leq \langle \lambda, \alpha_i^\vee \rangle \leq \ell - 1$ for all i . The general case can be easily deduced from the Steinberg tensor product theorem.

For any root α , the group G contains a subgroup isomorphic to (a quotient of) SL_2 . If $\langle \lambda, \alpha^\vee \rangle = n \leq \ell - 1$ then the action of SL_2 on the highest weight vector will give rise to the weights $\lambda, \lambda - \alpha, \dots, \lambda - n\alpha$. By examining the lengths of these weights we see that only the last one is in the Weyl group orbit of λ . This forces $n = 0$ or 1 if the representation is to be miniscule. In particular, if we write $n_i = \langle \lambda, \alpha_i^\vee \rangle$, then $n_i = 0$ or 1 for all i . Next, we claim that only one n_i could be 1 . Otherwise, we can pick a path in the Dynkin diagram $\alpha_i, \alpha_{i+1}, \dots, \alpha_j$ such that $n_i = n_j = 1$ and $n_k = 0$ for any α_k between α_i and α_j . Consider the root $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$. Since $\langle \lambda, \alpha^\vee \rangle = 2$, this is a contradiction. (Note that we have just used the condition $\ell \neq 2$.) Finally, if the weight λ is fundamental but not miniscule then there exists a positive root α such that $\langle \lambda, \alpha^\vee \rangle = 2$. Again, a contradiction.

The proof of (2) is similar, except when $\langle \lambda, \alpha^\vee \rangle = 2$. Then $\lambda, \lambda - \alpha$ and $\lambda - 2\alpha$ are weights with $\lambda - \alpha$ the shortest length among the three weights. Since 0 is the only other orbit of weights, we must have $\lambda = \alpha$. If α is a long highest root, and the root system is of the type B_n, C_n or F_4 then there exists a short root β such that $\langle \alpha, \beta^\vee \rangle = 2$. This implies that the short root $\alpha - \beta$ is also a weight, and the representation cannot be almost miniscule. If $\ell = 3$ and $G = G_2$ then this argument breaks down. The adjoint representation breaks up as $14 = 7 + 7'$ where 7 is the representation whose non-trivial weights are short roots, while $7'$ is a representation whose non-trivial weights are long roots. \square

We also note that the 26-dimensional almost miniscule representation of F_4 is not irreducible. It breaks up as $26 = 25 + 1$. The main result of this section is the following:

Proposition 7.2. *Let G be a connected reductive group over an algebraically closed field of characteristic different from 2. Let V be an irreducible and faithful representation of G of dimension $2n + 1$, preserving a non-degenerate bilinear form. Assume that there exist $2n$ different weights in V permuted cyclically by a Weyl group element. Then $G = \mathrm{SO}_{2n+1}$ or G_2 if $n = 3$.*

Proof. Let Z be the connected component of the center of G . The characters of Z form a lattice. In particular, the only self-dual character is the trivial character. This shows, since V is faithful, that Z is trivial. Therefore G is semi-simple. Since weights of a self-dual representation come in pairs $\{\mu, -\mu\}$, and the Weyl group preserves the length of weights, the weight outside the cycle must be trivial, so $\dim(V^T) = 1$. Next, let $G_1 \times \dots \times G_k$ be a product of almost simple groups isogenous to G such that $V = V_1 \otimes \dots \otimes V_k$ is a tensor product of irreducible, *non-trivial* representations of G_1, \dots, G_k . Since $V^T \neq 0$, the zero weight must appear in each V_1, \dots, V_k . But then V can be almost miniscule only if $k = 1$. From the list of almost miniscule representations of almost simple groups we see that V must be the standard representation of SO_{2n+1} or G_2 or a Frobenius twist of it, if $\ell \neq 3$. If $\ell = 3$ then F_4 has an almost miniscule representation of dimension 25. However, since F_4 has no element of order 24 in its Weyl group, there is no element permuting all non-trivial weights. This completes the proof. \square

8. GALOIS REPRESENTATION

Let Π be the self-dual cuspidal automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$ constructed by lifting from Sp_{2n} , or from $\mathrm{G}_2(\mathbb{A})$ in Section 6. In particular,

- The infinitesimal character of Π_∞ is regular and integral.
- The local component Π_v is unramified for all $v \neq \ell, q$
- The local parameter of Π_q is $\phi(\tau) \oplus \chi_q$ where q splits in any quadratic extension of \mathbb{Q} ramified at ℓ only.
- If $\ell \neq 2$ then Π_2 is unramified or, if $G = \mathrm{Sp}_{2n}$, the local parameter is the irreducible representation ϕ_2 .

By the generalization of Theorem 1.1 due to Shin, and the remark after it, one can attach a semi-simple Galois representation to Π :

$$r_\Pi : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_{2n+1}(\bar{\mathbb{Q}}_\ell)$$

such that for every prime $v \neq \ell$ the restriction of r to the decomposition group D_v gives the Langlands parameter of Π_v , up to a Frobenius semi-simplification.

[Note that when the Weil-Deligne parameter $\mathcal{L}(\Pi_v)$ for all $v \neq \ell$ has monodromy $N = 0$, we can state Theorem 1.1 as :

$$r_\Pi|_{D_v}^{\mathrm{Frob-ss}} = \mathcal{L}(\Pi_v)$$

and where $\mathcal{L}(\Pi_v)$ may be regarded, using the embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$, as a representation of the decomposition group D_v at v with values in $\mathrm{GL}_{2n+1}(\bar{\mathbb{Q}}_\ell)$.]

Let us concentrate on the prime q . Here Π_q is the lift of a supercuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{Q}_q)$ whose parameter, when restricted to the inertia group $I_{\mathbb{Q}_q}$, is a direct sum of $2n + 1$ one dimensional characters. One trivial and $2n$ other cyclically permuted by Fr_q . It follows that $r_\Pi(\mathrm{Fr}_q^{2n})$ commutes with $r_\Pi(I_q)$. This shows that $r_\Pi(\mathrm{Fr}_q^{2n})$ and $r_\Pi(\mathrm{Fr}_q)$ must be semi-simple. In other words, r gives exactly the parameter of Π_q . The same argument shows that the restriction of r_Π to D_2 gives the parameter of Π_2 if Π_2 is supercuspidal.

Proposition 8.1. *If Π satisfies the above conditions at places $\infty, 2$ and q then the Galois representation r_Π is irreducible and orthogonal.*

Proof. If Π_2 is supercuspidal, then the local parameter is irreducible and r_Π is irreducible. Thus, assume that Π_2 is unramified (or $\ell = 2$). If r_Π is reducible then r_Π has two irreducible summands of dimensions $2n$ and 1 . Since the eigenvalues of $r_\Pi(\mathrm{Fr}_v)$ are $1, \lambda_i^\pm, (1 \leq i \leq n)$ the representation r_Π is self-dual. In particular, the one-dimensional summand is a quadratic character χ ramified at ℓ only and such that $\chi(\mathrm{Fr}_q) = -1$. This implies that there exists a quadratic extension ramified at ℓ only and such that q stays inert. This is a contradiction since q is picked so that it splits completely in every quadratic extension ramified at ℓ only. Therefore r_Π is irreducible. Since it is self-dual and of odd dimension it is orthogonal as well. \square

9. ZARISKI CLOSURE

We shall now assume that the odd prime q and the parameter $\phi(\tau) \oplus \chi_q$ of the local component Π_q of Π are picked using Lemma 3.3. In particular, the representation r_Π is unramified at all primes different from $2, \ell$ and q , and $r_\Pi(I_q)$, the image of the inertia

subgroup is of order p . Let $\Gamma = r_{\Pi}(G_{\mathbb{Q}})$. If $d > 1$ is an integer, let Γ^d be the intersection of all normal subgroups of Γ of index $\leq d$. The following is crucial:

Lemma 9.1. *Assume that the local component Π_q is constructed by means of Lemma 3.3. Then $r_{\Pi}(D_q)$ is contained in Γ^d .*

Proof. Let Γ' be a normal subgroup of index $\leq d$. We must show that the image of D_q lands in Γ' . Let L be the Galois extension of \mathbb{Q} corresponding to Γ' . Obviously, L is unramified at all primes except $2, \ell$ and q . Moreover, since $r_{\Pi}(I_q)$ is of order p and $p > d$, it follows that L is unramified at q as well. Thus, L is contained in K , the compositum of all Galois extensions ramified at $2, \ell$ and of degree $\leq d$. This implies that q splits completely in L (by Lemma 3.3) and $r_{\Pi}(D_q)$ is therefore contained in Γ' , as desired. \square

The above lemma shows that if Π_q is constructed by means of Lemma 3.3 then $\Gamma_{2n,p}$, the image of the decomposition group D_q , sits *deeply embedded* in $\Gamma = r_{\Pi}(G_{\mathbb{Q}})$. This property is crucial in controlling the size of the image Γ of the Galois group.

Theorem 9.2. *There exists a function $I : \mathbb{N} \rightarrow \mathbb{N}$ such that if Π is a self-dual cuspidal automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$, as in Section 8, and such that the local parameter of Π_q is constructed by means of Lemma 3.3 with $d > I(n)$ then the Zariski closure of $r_{\Pi}(G_{\mathbb{Q}})$ is isomorphic to $\mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_{\ell})$ if $n \neq 3$ and $\mathrm{SO}_7(\overline{\mathbb{Q}}_{\ell})$ or $\mathrm{G}_2(\overline{\mathbb{Q}}_{\ell})$ if $n = 3$.*

Proof. Let G be the Zariski closure of $r_{\Pi}(G_{\mathbb{Q}})$. Since r_{Π} is irreducible this is a reductive group. Let G° be its connected component. Recall that the image of the inertia subgroup I_q contains an element s in $\mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_{\ell})$ of order p . We want to show that s is contained in G° . We need the following (Lemma 6.3 in [KLS]).

Lemma 9.3. *There exists a function $J : \mathbb{N} \rightarrow \mathbb{N}$ such that every integer $n > 0$ and every algebraic subgroup $G \subset \mathrm{GL}_n$ over a field of characteristic zero, there is normal subgroup $G_1 \subseteq G$ of index $\leq J(n)$ containing G° such that G_1/G° is abelian.*

If we pick $d > J(n)$ then, by Lemma 9.1, the image of D_q must be contained in G_1 . Since $\Gamma_{2n,p}$ is a semi-direct product of $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/2n\mathbb{Z} \subset \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/(p-1)\mathbb{Z}$ one easily sees that the commutator subgroup of $r_{\Pi}(D_q)$ is $r_{\Pi}(I_q) \cong \mathbb{Z}/p\mathbb{Z}$. This shows that the projection of $r_{\Pi}(D_q)$ to the abelian quotient G_1/G° must contain s in the kernel. In other words, s is in G° .

Recall that the eigenvalues of s are

$$\tau, \tau^q, \dots, \tau^{q^{n-1}}, \tau^{-1}, \tau^{-q}, \dots, \tau^{-q^{n-1}}, 1.$$

where τ is a p th root of one. Moreover, these eigenvalues are *distinct*. It follows that the centralizer of s in GL_{2n+1} is a torus. Thus, s is a regular semi-simple element in G° and the centralizer of s in G° is a maximal torus T . Next, note that the centralizer of s in G° is the same as the centralizer of $r_{\Pi}(\mathrm{Fr}_q) \cdot s \cdot r_{\Pi}(\mathrm{Fr}_q^{-1})$ (since it is so in GL_{2n+1}). It follows that $r_{\Pi}(\mathrm{Fr}_q)$ normalizes T . We need the following lemma.

Lemma 9.4. *There exists a function $J' : \mathbb{N} \rightarrow \mathbb{N}$ such that every integer $n > 0$ and every reductive group G of rank n over an algebraically closed field of characteristic zero, there is normal subgroup $G_1 \subseteq G$ of index $\leq J'(n)$ containing G° such that conjugation of G° by any element in G_1 is inner.*

Proof. The conjugation action of G on G° gives a homomorphism from a finite group G/G° to $\text{Aut}(G^\circ)/\text{Inn}(G^\circ)$. The latter group is contained in the group of automorphisms of a quadruple $(X, \Delta, X^\vee, \Delta^\vee)$ where X and X^\vee are the character and co-character lattices of a maximal torus T in G° , and Δ a choice of simple roots for G° . It follows that G/G° maps into $\text{GL}(X) \cong \text{GL}_n(\mathbb{Z})$. Since there is a torsion-free congruence subgroup in $\text{GL}_n(\mathbb{Z})$, the order of any finite subgroup in $\text{GL}_n(\mathbb{Z})$ is bounded by the index of the congruence subgroup. This proves the lemma. \square

Now, if we pick $d > J'(n)$ then $r_\Pi(D_q)$ must be contained in G_1 . This shows that the normalizer of the torus T in G° contains an element of order $2n$ which cyclically permutes the weights corresponding to the $2n$ eigenvalues of s different from 1. This shows that r_Π , under the action of G° , has at most two irreducible summands, of dimension $2n$ and 1. In fact, since G° is a normal subgroup and G acts irreducibly, a simple argument shows that G° must act irreducibly as well. It follows that G° satisfies conditions of Proposition 7.2, so it must be either SO_{2n+1} or G_2 . It remains to show that $G = G^\circ$. This is easy. First, r_Π is an irreducible $2n + 1$ -dimensional representation of G which restricts to the standard representation of G° . Second, since SO_{2n+1} and G_2 have no outer automorphisms, every connected component of G contains an element commuting with G° . Combining the two, it follows that any connected component of G contains a homothety by a scalar λ . On the other hand, since $r_\Pi(\text{Fr}_v)$ has 1 as one of the eigenvalues for almost all primes v , by Čebotarev's density the function $\det(1 - r_\Pi(g))$ must be 0 for all elements g in G . In particular, λ must be equal to 1 and this shows that $G = G^\circ$. The proposition is proved with $I(n) = \max(J(n), J'(n))$. \square

According to Theorem 9.2 there are two possibilities for the Zariski closure if $n = 3$. The following two corollaries give us more precise statements in this case.

Corollary 9.5. *Assume that Π comes from $G_2(\mathbb{A})$. Then the Zariski closure of $r_\Pi(G_\mathbb{Q})$ is $G_2(\overline{\mathbb{Q}}_\ell)$.*

Proof. Let G be the Zariski closure. We know that G is either SO_7 or G_2 . It suffices to show that the rank of the maximal torus T is at most 2. To see this, recall that the eigenvalues of $r_\Pi(\text{Fr}_v)$ are $1, \lambda_i^\pm$ ($i = 1, 2, 3$) for $v \neq q, \ell$. Therefore the characteristic polynomial of $r_\Pi(\text{Fr}_v)$ is $f(x) = (x - 1)g(x)$ with

$$g(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1$$

a palindromic polynomial of degree 6. Moreover, the condition $\lambda_1 \lambda_2 \lambda_3 = 1$ gives one algebraic relation on the three coefficients a, b and c . By Čebotarev's density theorem, the same holds for the characteristic polynomial of all elements in the image of r and, therefore, for all elements in T . In particular, the dimension of T is less than or equal to 2. \square

Corollary 9.6. *Assume that Π_2 , the local component of Π , is the irreducible self-dual cuspidal representation of $\text{GL}_7(\mathbb{Q}_2)$ introduced in Section 4. Then the Zariski closure of r_Π is $\text{SO}_7(\overline{\mathbb{Q}}_\ell)$.*

Proof. This is easy since the image of the inertia subgroup I_2 contains elements in $\text{SO}_7(\overline{\mathbb{Q}}_\ell)$ which are not contained in G_2 . \square

10. A GROUP-THEORETIC CRITERION

In this section we develop certain criteria which give us control over the image of ℓ -adic representations in the case of exceptional groups. As such, this section is somewhat more general than what is needed for the main results in this paper. However, the results of this section might have future applications. A possibility in this direction is presented in Section 12.

Let Γ be a profinite group and $d \geq 2$ an integer. We define Γ^d as the intersection of all open normal subgroups of Γ of index $\leq d$.

Lemma 10.1. *If Γ is a profinite group, Δ a closed normal subgroup, and d a positive integer, then the image of Γ^d in Γ/Δ is $(\Gamma/\Delta)^d$.*

Proof. The image in Γ/Δ of every open subgroup of Γ of index $\leq d$ is again open of index $\leq d$, and conversely, all open index $\leq d$ subgroups of Γ/Δ arise as images of open index $\leq d$ subgroups of Γ . \square

Let $n \geq 2$ be an integer and p a prime congruent to 1 (mod n).

Definition 10.2. *By a group of type (n, p) , we mean any finite group Γ with a normal subgroup Δ isomorphic to $\mathbb{Z}/p\mathbb{Z}$ such that the image of $\text{Inn}\Gamma$ in $\text{Aut}\Delta$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

As noted in the introduction, this is slightly different from the terminology in [KLS]. If $\ell \neq p$ is prime, a group of type (n, p, ℓ) will mean a (possibly finite) profinite group which is the extension of a group of type (n, p) by a pro- ℓ group.

Lemma 10.3. *If $0 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3 \rightarrow 0$ is a short exact sequence of profinite groups, $n \geq 2$, ℓ and p are distinct primes, and Γ_1 is pro- ℓ , then Γ_2 contains a subgroup of type (n, p, ℓ) if and only if Γ_3 does.*

Proof. Any extension of a group of type (n, p, ℓ) by a pro- ℓ group is again of type (n, p, ℓ) , so one direction is trivial.

For the other, let Δ_2 be a closed subgroup of Γ_2 and Δ'_2 an open pro- ℓ subgroup of Δ_2 such that $\Delta''_2 := \Delta_2/\Delta'_2$ is of type (n, p) . Let $\Delta_1 := \Delta_2 \cap \Gamma_1$, $\Delta'_1 := \Delta'_2 \cap \Gamma_1$, and $\Delta''_1 := \Delta_1/\Delta'_1$. By the snake lemma, we have a right-exact sequence

$$\Delta'_2/\Delta'_1 \rightarrow \Delta_2/\Delta_1 \rightarrow \text{coker}(\Delta''_1 \rightarrow \Delta''_2) \rightarrow 0.$$

As Δ'_2 is pro- ℓ , so is every quotient thereof, so Δ_2/Δ_1 is the extension of $\text{coker}(\Delta''_1 \rightarrow \Delta''_2)$ by a pro- ℓ group.

Every quotient of a group $\Gamma_{n,p}$ of type (n, p) by an ℓ -group is again of type (n, p) . Indeed, the quotient map preserves the normal subgroup of $\Gamma_{n,p}$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and therefore the image of $\text{Inn}\Gamma_{n,p} \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$. \square

We remark that for the non-trivial direction, the proof uses only the fact that Γ_1 is the inverse limit of finite groups of prime-to- p order.

Theorem 10.4. *Let ℓ be a prime and G a connected reductive algebraic group over $\bar{\mathbb{F}}_\ell$. There exists an absolute constant B such that*

- (1) If $\text{rk } G \leq 2$ and $p > B$ is a prime distinct from ℓ and $G(\overline{\mathbb{F}}_\ell)$ contains a subgroup of type $(6, p, \ell)$, then G is of type G_2 .
- (2) If $\text{rk } G \leq 4$ and $p_1, p_2 > B$ are primes distinct from ℓ and $G(\overline{\mathbb{F}}_\ell)$ contains subgroups of type $(8, p_1, \ell)$ and $(12, p_2, \ell)$, then G is of type F_4 .
- (3) If $\text{rk } G \leq 6$ and $p > B$ is a prime distinct from ℓ and $G(\overline{\mathbb{F}}_\ell)$ contains a subgroup of type $(9, p, \ell)$, then G is of type E_6 .
- (4) If $\text{rk } G \leq 7$ and $p_1, p_2 > B$ are primes distinct from ℓ and $G(\overline{\mathbb{F}}_\ell)$ contains subgroups of type $(18, p_1, \ell)$ and $(30, p_2, \ell)$, then G is of type E_7 .
- (5) If $\text{rk } G \leq 8$ and $p_1, p_2, p_3 > B$ are primes distinct from ℓ and $G(\overline{\mathbb{F}}_\ell)$ contains subgroups of type $(18, p_1, \ell)$, $(20, p_2, \ell)$, and $(30, p_3, \ell)$, then G is of type E_8 .

Proof. We claim that the root systems of type G_2 , F_4 , E_6 , E_7 , and E_8 respectively are the only root systems of rank less than or equal to 2, 4, 6, 7, and 8 respectively, which have Weyl group elements of order 6; 8 and 12; 9; 18 and 30; and 18, 20, and 30 respectively. To see that this is so, we compile a table of the orders of Weyl group elements for root systems of rank ≤ 8 . We write each root system as a sum of irreducible root systems, each coded as a single letter and a single digit, and arranged alphabetically. We order root systems first by rank and within rank, alphabetically. A root system is *exceptional* if it is simple of type G_2 , F_4 , E_6 , E_7 , or E_8 . For brevity, we omit those root systems for which the set of possible Weyl element orders is a proper subset of that for some non-exceptional root system of equal or inferior rank. In case of equality, we print only the lexicographically smallest example. (For example, in our table, no root system of type C_n appears, since B_n is lexicographically inferior to it and has the same set of Weyl group orders. Likewise, $A_1 + A_4$ does not appear because its set of Weyl group orders is strictly dominated by that of B_5 , which, though lexicographically superior, has the same rank.) Since the set of orders of elements of a finite group is determined by its subset of maximal elements with respect to divisibility, we exhibit only this subset.

Root System	Maximal Elements
A1	2
A2	2,3
B2	4
G2	6
B3	4,6
A2+B2	12
A4	4,5,6
B4	4,6,8
F4	8,12
B5	8,10,12
A2+B4	24
A4+B2	12,20
A4+G2	12,30
A6	7,10,12
E6	8,9,10,12

A1+E6	8,10,12,18
A2+B5	24,30
A4+B3	12,20,30
A7	7,8,10,12,15
B7	14,20,24
E7	8,12,14,18,30
A2+E6	18,24,30
A4+F4	24,40,60
A6+B2	12,20,28
A6+G2	12,30,42
A8	8,9,12,14,15,20
B2+E6	8,20,36
B8	14,16,20,24,30
E8	14,18,20,24,30

Now, let $\Gamma_{n,p,\ell}$ be a subgroup of $G(\overline{\mathbb{F}}_\ell)$ of type (n, p, ℓ) for some $n \geq 2$ and some prime $p \neq \ell$. Let $\Delta_\ell \subset \Gamma_{n,p,\ell}$ denote a normal ℓ -subgroup such that the corresponding quotient group $\Gamma_{n,p}$ is of type (n, p) . Now, $\Delta_\ell \subset G(\mathbb{F}_{\ell^k})$ for some k , and so it is contained in a Sylow ℓ -subgroup of $G(\mathbb{F}_{\ell^k})$. Such a subgroup is the group of \mathbb{F}_{ℓ^k} -points of the unipotent radical of a Borel subgroup of G . By [Hu1, 30.3], the normalizer of Δ_ℓ in G is a parabolic subgroup P , proper if Δ_ℓ is non-trivial. If N denotes the unipotent radical of P , then $N(\overline{\mathbb{F}}_\ell) \cap \Gamma_{n,p,\ell}$ is an ℓ -group, so by Lemma 10.3, the image of $\Gamma_{n,p,\ell}$ in the Levi factor $M(\overline{\mathbb{F}}_\ell)$ is again of type (n, p, ℓ) . The rank of M is equal to that of G , while the dimension is strictly less. Iterating this process we end up with a connected reductive group, which we again denote M , of the same rank as G such that $M(\overline{\mathbb{F}}_\ell)$ contains a subgroup $\Gamma_{n,p}$ of type (n, p) . If M is an exceptional group, then $G = M$, since in each rank r , the exceptional group, if one exists, is the connected reductive group of maximal dimension in rank r .

Let $\Gamma_{n,p}$ be a subgroup of type (n, p) of $M(\overline{\mathbb{F}}_\ell)$ for some integer n , let x denote the image of a generator of the normal subgroup of $\Gamma_{n,p}$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and let $a \in \mathbb{Z}$ be such that its image in $\mathbb{Z}/p\mathbb{Z}$ is of order n in $(\mathbb{Z}/p\mathbb{Z})^\times$. As x and x^a are conjugate in $\Gamma_{n,p}$, x and x^a are conjugate in $M(\overline{\mathbb{F}}_\ell)$. Since $p \neq \ell$, they are semisimple elements, and if $B < p$ is taken to be larger than the maximal number of components of the centralizer of any semisimple element in any reductive connected group of rank ≤ 8 , it follows that x and x^a belong to a common maximal torus $T \subset M$. By a well-known theorem [Hu2, §3.1], there exists $w \in N_M(T)(\overline{\mathbb{F}}_\ell)$ such that $wxw^{-1} = x^a$. However, this implies that the order of the image of w in the Weyl group of M with respect to T is divisible by n . From our analysis of orders of elements in Weyl groups in rank ≤ 8 , it follows that M is exceptional and therefore that G is exceptional. \square

Theorem 10.5. *There exist constants A and B such that if*

- (1) $d > A$ is an integer,
- (2) $p_1, p_2, p_3 > B$ and $\ell \notin \{p_1, p_2, p_3\}$ are primes,
- (3) K is an ℓ -adic field,
- (4) G is a connected reductive algebraic group over K such that
 - (a) $\text{rk } G \leq 2$,
 - (b) $\text{rk } G \leq 4$,

- (c) $\text{rk } G \leq 6$,
 - (d) $\text{rk } G \leq 7$, or
 - (e) $\text{rk } G \leq 8$,
- (5) $\Gamma \subset G(K)$ is a profinite subgroup such that (respectively)
- (a) Γ^d has a subgroup of type $(6, p_1, \ell)$;
 - (b) Γ^d has a subgroup of type $(8, p_1, \ell)$ and $(12, p_2, \ell)$;
 - (c) Γ^d has a subgroup of type $(9, p_1, \ell)$;
 - (d) Γ^d has subgroups of type $(18, p_1, \ell)$ and $(30, p_2, \ell)$; or
 - (e) Γ^d has subgroups of type $(18, p_1, \ell)$, $(20, p_2, \ell)$, and $(30, p_3, \ell)$,

then some finite quotient $\bar{\Gamma}$ of Γ satisfies

$$(10.6) \quad (H^{\text{ad}}(\bar{\mathbb{F}}_\ell)^F)^{\text{der}} \subset \bar{\Gamma} \subset H^{\text{ad}}(\bar{\mathbb{F}}_\ell)^F,$$

where F is a Frobenius map and H^{ad} is a simple adjoint algebraic group of type G_2, F_4, E_6, E_7 , or E_8 respectively. In particular, in the first, second, and fifth case, $\bar{\Gamma}$ is simple.

Proof. Replacing K by a finite extension, we may assume first that G is split, and second that Γ fixes a hyperspecial vertex of the building of G over K (see, e.g., [La], and the remark below). Thus, there exists a smooth group scheme \mathcal{G} over the ring of integers \mathcal{O} of K whose generic fiber is G , whose special fiber is again reductive and connected, and such that $\Gamma \subset \mathcal{G}(\mathcal{O})$. The root datum of the special fiber of \mathcal{G} is the same as that of G . Let $H := \mathcal{G}_{\bar{\mathbb{F}}_\ell}$ denote the geometric special fiber. Thus, Γ maps to $H(\bar{\mathbb{F}}_\ell)$ with finite image and pro- ℓ kernel. Replacing Γ by its image in $H(\bar{\mathbb{F}}_\ell)$, by Lemma 10.1 and Lemma 10.3, we still have that Γ^d has subgroups of the specified types. By Theorem 10.4, H is almost simple of type G_2, F_4, E_6, E_7 , or E_8 according as we are in case (a), (b), (c), (d), or (e). Assuming $p > 3$, the image $\bar{\Gamma}$ of Γ in $H^{\text{ad}}(\bar{\mathbb{F}}_\ell)$ again has the property that $\bar{\Gamma}^d$ has subgroups of the specified types. By [LP, Th. 0.5], it follows that either $\bar{\Gamma}$ satisfies the condition (10.6) for some Frobenius map or that $\bar{\Gamma}$ is contained in a proper algebraic subgroup of $I \subset H^{\text{ad}}$ with a component group I/I° whose order is bounded above by an absolute constant A . As $d > A$, it follows that $\bar{\Gamma}^d \subset I^\circ(\bar{\mathbb{F}}_\ell)$. If N denotes the unipotent radical of I° , the image of $\bar{\Gamma}^d$ in $(I/N)(\bar{\mathbb{F}}_\ell)$ contains a subgroup of type (n, p, ℓ) . As I/N is reductive of rank less than or equal to the rank r of H^{ad} (which equals the rank of G) and as $\dim I/N \leq \dim I < \dim H^{\text{ad}} = \dim G$, it follows that I/N cannot be exceptional of rank r , which contradicts Theorem 10.4. \square

Remark: If G is of type G_2, F_4 or E_8 then $G(K)$ is always split and simply connected. The profinite subgroup Γ is contained in a maximal parahoric subgroup of $G(K)$. The quotient of this maximal parahoric subgroup by its pro- ℓ radical is a simply connected semisimple group H of rank r over the residual field of K . Let $\bar{\Gamma}$ be the projection of Γ into H . If Γ contains groups of type (n, p, ℓ) as specified in Theorem 10.5, then so does $\bar{\Gamma}$. By Theorem 10.4 H must be of the same type as G . This shows that the maximal parahoric subgroup containing Γ is hyperspecial.

11. MAIN THEOREM

We are now ready to construct finite Galois groups over \mathbb{Q} . Let $r_\Pi : G_{\mathbb{Q}} \rightarrow G(\bar{\mathbb{Q}}_\ell)$ be the Galois representation attached to a self-dual cuspidal representation Π constructed in Section

6. Recall that Π is constructed so that the image of D_q in $\Gamma = r_\Pi(G_\mathbb{Q})$ is a group of type $(2n, p)$, denoted by $\Gamma_{2n,p}$, and contained in Γ^d . By Theorem 9.2 and its corollaries, if $d > I(n)$ then $G = G_2$ if Π is a lift from G_2 and $G = \mathrm{SO}_{2n+1}$ if Π is a lift from Sp_{2n} and the local component Π_2 is the supercuspidal representation defined in Section 4. In particular, $\ell > 2$ if $G = \mathrm{SO}_{2n+1}$. If $G = G_2$ then by Theorem 10.5 Γ has a quotient isomorphic to $G_2(\mathbb{F}_{\ell^k})$ (or a Ree group if $\ell = 3$) for some k provided that $d > A, B$ where A and B are in the statement of Theorem 10.5.

Assume now that $G = \mathrm{SO}_{2n+1}$ and that Π is a lift from Sp_{2n} and the local component Π_2 is the supercuspidal representation defined in Section 4.

Theorem 11.1. *Assume that $\ell > 2$. There exists a function $d : \mathbb{N} \rightarrow \mathbb{N}$ such that if Π_q is picked with $d > d(n)$ then $\Gamma = r_\Pi(G_\mathbb{Q})$ has a quotient $\bar{\Gamma}$ such that*

$$(11.2) \quad (\mathrm{SO}_{2n+1}(\bar{\mathbb{F}}_\ell)^F)^{\mathrm{der}} \subset \bar{\Gamma} \subset \mathrm{SO}_{2n+1}(\bar{\mathbb{F}}_\ell)^F,$$

where F is a Frobenius map.

Proof. By enlarging the field K we can assume that G is split and that Γ is contained in a hyperspecial maximal parahoric subgroup. This means that G can be written as $G = \mathrm{SO}(V, Q)$ where V is a linear space over K and Q a split quadratic form, and there exists a lattice L stabilized by Γ such that Q takes integral values on L . Moreover, if \bar{V} is the reduction modulo ℓ of L , then the quadratic form Q reduces modulo ℓ to a non-degenerate quadratic form \bar{Q} on \bar{V} . In other words, the pair (L, Q) defines a smooth group scheme over the ring of integers of K whose generic fiber is G and such that $H := \mathrm{SO}(\bar{V}, \bar{Q})$ is the special fiber. Let $\bar{\Gamma}$ be the image of Γ in $H(\bar{\mathbb{F}}_\ell)$. By [LP, Th. 0.5], it follows that either $\bar{\Gamma}$ satisfies the condition (11.2) for some Frobenius map or that $\bar{\Gamma}$ is contained in a proper algebraic subgroup of $I \subset H$ with a component group I/I° whose order is bounded above by an absolute constant $A(n)$. If we pick $d > A(n)$ then $\bar{\Gamma}^d \subset I^\circ(\bar{\mathbb{F}}_\ell)$. It follows that $\Gamma_{2n,p}$ is contained in I° . Under the action of $\Gamma_{2n,p}$, the orthogonal space \bar{V} decomposes as a sum of two irreducible mutually orthogonal representations of dimensions $2n$ and 1 . This implies that the nilpotent radical of I° is trivial. Indeed, if N is a non-trivial unipotent radical of I° , then there exists a non-trivial subspace \bar{U} of \bar{V} fixed by N . Since $\Gamma_{2n,p}$ normalizes N , \bar{U} must be one of the two mutually orthogonal $\Gamma_{2n,p}$ -summands. By orthogonality, N must preserve the other summand. Since that summand is also $\Gamma_{2n,p}$ -irreducible, N must be trivial, a contradiction. Thus, I° is reductive. We claim that \bar{V} is an irreducible I° -module. If not, then I° admits a $2n$ -dimensional orthogonal representation such that there exists a Weyl group element in I° permuting transitively all weights. We need the following:

Lemma 11.3. *Let G be a connected reductive group over an algebraically closed field of characteristic $\neq 2$. Then G has no irreducible orthogonal representations of even dimension such that there exists a Weyl group element permuting all weights.*

Proof. Of course, we can assume that G is semi-simple and that $G = G_1 \times \cdots \times G_k$, a product of almost simple groups. If V is a miniscule, self-dual representation of G then $V = V_1 \otimes \cdots \otimes V_k$ where V_i are miniscule and self-dual representations of G_i . Of course, if G contains a Weyl group element permuting all weights in V , then the same holds for all pairs (G_i, V_i) . The converse is true only if the dimensions of V_i are pairwise relatively prime. If G is simple, an argument in [KLS] shows that the only miniscule, self-dual representations with such Weyl

group element is the standard representation of Sp_{2n} and its Frobenius twists. (The list there includes also the standard representation of SO_{2n} , but that one can be excluded by a direct inspection.) Since even numbers are never pairwise relatively prime, we can conclude that G is simple and that the representation is the standard representation of Sp_{2n} . This one is not orthogonal, however, and the lemma follows. \square

The lemma implies that \bar{V} is an irreducible representation of I° . Therefore, the pair (I°, \bar{V}) satisfies conditions of Lemma 7.2. It follows that $I^\circ = H$ or, if $n = 3$, $I^\circ = \mathrm{G}_2(\bar{\mathbb{F}}_\ell)$. Since G_2 has the trivial center, no outer automorphisms and it acts irreducibly on \bar{V} , any element of I/I° is represented by a scalar matrix. However, by Čebotarev's density theorem, any element in $\bar{\Gamma}$ has 1 as an eigenvalue. It follows that $I = I^\circ$. Since the image of the local decomposition group D_2 contains elements of order 2 which are not contained in G_2 we see that I cannot be G_2 . Thus we have $I^\circ = H$ in all cases. This is a contradiction. The theorem is proved with $d(n) = \max(A(n), I(n))$. \square

Summarizing, we have shown that mod ℓ reduction of the representations r_Π give rise to $\mathrm{G}_2(\mathbb{F}_{\ell^k})$ (or a Ree group if $\ell = 3$), $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ or $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$ as Galois group. In other words we have essentially proved the following theorem that we stated in the introduction:

Theorem 11.4. *Let t be a positive integer. We take t to be even if $\ell = 3$ in the first case below.*

- (1) *Let ℓ be a prime. Then there exists an integer k divisible by t such that the simple group $\mathrm{G}_2(\mathbb{F}_{\ell^k})$ appears as a Galois group over \mathbb{Q} .*
- (2) *Let ℓ be an odd prime. Then there exists an integer k divisible by t such that the finite simple group $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ or the finite classical group $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$ appears as a Galois group over \mathbb{Q} .*
- (3) *If $\ell \equiv 3, 5 \pmod{8}$, then there exists an integer k divisible by t such that the finite simple group $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ appears as a Galois group over \mathbb{Q} .*

Proof. The divisibility of k by t follows from part (3) of Lemma 3.3. The remark following Lemma 3.3 shows that by taking t even if $\ell = 3$ in the first case eliminates Ree groups. It remains to deal with the third statement. Assume now that the Galois group given by part (2) is $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$. Then the subgroup $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ of index 2 defines a quadratic field K . This field is unramified for all primes different from $2, \ell$ and q , since the same holds for the representation r_Π . Since the image of the inertia I_q is of order p , it lands in the subgroup $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$. Thus, K is unramified at q also. Moreover, by Proposition 4.1, the image of the decomposition group D_2 is a group such that the quotient by its commutator is odd. Such group must be contained in the subgroup $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$. This shows not only that K is unramified at 2 but 2 splits in K . We remind the reader that the unique quadratic field ramified at ℓ and no other primes is $\mathbb{Q}(\sqrt{\ell})$ if $\ell \equiv 1 \pmod{4}$ and $\mathbb{Q}(\sqrt{-\ell})$ if $\ell \equiv 3 \pmod{4}$. However, since 2 splits in this field if and only if $\ell \equiv 1, 7 \pmod{8}$ we see that if $\ell \equiv 3, 5 \pmod{8}$ the Galois group constructed in part (2) is in fact $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$. \square

12. ON FUTURE DIRECTIONS

One difficulty that we needed to address in this paper came from the fact that the group $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$ has no self-dual supercuspidal representation unless $p = 2$. In order to construct

Galois groups of type B_n this problem was resolved by introducing the self-dual supercuspidal representation Π_2 of $GL_{2n+1}(\mathbb{Q}_2)$ whose parameter contains a Jordan subgroup of $SO_{2n+1}(\mathbb{C})$. For Galois groups of type G_2 the construction is based on a technical improvement of Theorem 1.1 due to Shin, which is based on the fundamental lemma for unitary group. Another way, which avoids the use of the fundamental lemma, would be to pick Π_2 so that its parameter comes from the Jordan subgroup in G_2 . More precisely the parameter of Π_2 should be the homomorphism $\phi_2 : W_{\mathbb{Q}_2} \rightarrow G_2$ described in Proposition 4.2. In order to obtain a global lift from G_2 to GL_7 with this Π_2 as a local component one would need to complete the following (doable) program:

- Define, via induction from open compact subgroup, a generic supercuspidal σ_2 representation of $G_2(\mathbb{Q}_2)$ corresponding to the parameter ϕ_2 .
- Compute the theta lift of σ_2 to $PGSp_6(\mathbb{Q}_2)$.
- Show, using the method of [Sa2], that the further lift to $GL_7(\mathbb{Q}_2)$ is Π_2 .

A construction of supercuspidal representations attached to parameters arising from Jordan subgroups is a subject of the forthcoming paper by Gross and Reeder [GR]. The completion of the three step program would give Galois groups of type G_2 except in the residual characteristic 2 without using results of Shin [Sh]. There is yet another approach which removes the restriction $\ell \neq 2$, but introduces a different conjecture. Let G denote Sp_{2n} or G_2 . Then using the trace formula it is possible to show that there exist a cuspidal automorphic representation σ of $G(\mathbb{A})$ unramified at all primes different from ℓ, q and such that

- σ_∞ is a discrete series representation with a large (unspecified) parameter (weight).
- σ_q is a specified supercuspidal representation.
- σ_ℓ is the Steinberg representation.

Assuming that σ is globally generic then Π , the lift of σ to $GL_{2n+1}(\mathbb{A})$, is automatically cuspidal and has a discrete series representation at one local place, since Π_ℓ is the Steinberg representation of $GL_{2n+1}(\mathbb{Q}_\ell)$. (We use here that the theta lift of the Steinberg representation of $G_2(\mathbb{Q}_\ell)$ is the Steinberg representation of $PGSp_6(\mathbb{Q}_\ell)$, see [GrS].) Since matrix coefficients of the Steinberg representation are in $L^{1+\epsilon}(G)$ and therefore not integrable, we note that the method of Poincaré series cannot be used to construct such σ .

In principle our method could be extended to other groups. The main limitation at the moment is the lack of ℓ -adic representations attached to automorphic representations. If we assume, for example, that one can attach a 26-dimensional ℓ -adic representation to an algebraic automorphic form of the exceptional group F_4 then we would be able to construct finite groups of type F_4 as Galois groups over \mathbb{Q} . Indeed, to this end one would pick a cuspidal automorphic form σ of F_4 such that for two primes p_1 and p_2 the local components σ_{p_1} and σ_{p_2} are tame supercuspidal representations whose parameters have groups of type $(8, p_1)$ and $(12, p_2)$, respectively, as the image. Thus, if the two parameters at p_1 and p_2 are picked so that the images of the local decomposition groups are deeply embedded, then the results of Section 10 imply that the restriction modulo ℓ of the ℓ -adic representation attached to σ will give finite groups of type $F_4(\ell^k)$ as Galois groups over \mathbb{Q} .

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