1) Compute $3^{25}$ modulo 45. Hint: compute $3^{25}$ modulo 9 and 5, then use CRT.

2) Solve the congruence

$$x^5 \equiv 3 \pmod{64}.$$ 

3) Let $g$ be a group element such that $g^9 = e$ and $g^{16} = e$ where $e$ is the identity element. Show that $g = e$.

4) Let $R$ be a ring. Let 0 and 1 denote the identity elements for addition and multiplication, respectively. For every $r \in R$ prove that

- $r \cdot 0 = 0$.
- $(-1) \cdot r$ is the inverse of $r$ with respect to addition.

5) Calculate orders of all non-zero elements modulo 13.

6) Calculate $\Phi_{12}$ in two ways:

   a) By factoring $x^{12} - 1 = 0$, and using $\Phi_d(x)$ for all proper divisors $d$ of 12.

   b) Using primitive roots modulo 13, computed in the previous exercise, to calculate $\Phi_{12}(x)$ explicitly.

7) Use Lucas-Lehmer test to show that $M_{11} = 2^{11} - 1$ is not prime.

8) Let $n$ be a positive integer. Let $p$ be a prime divisor of $n^2 + 3$ i.e. $n^2 \equiv -3 \pmod{p}$. Use

$$\left(\frac{n^2}{p}\right) = \left(\frac{-3}{p}\right)$$

and quadratic reciprocity to conclude that $p \equiv 1 \pmod{3}$.

9) Let $M_\ell = 2^\ell - 1$ be a Mersenne number. Show that $M_\ell \equiv 1 \pmod{7}$ if $\ell \equiv 1 \pmod{3}$ and $M_\ell \equiv 3 \pmod{7}$ if $\ell \equiv 2 \pmod{3}$.

10) Let $M_\ell = 2^\ell - 1$ be a Mersenne number where $\ell \equiv 1 \pmod{3}$. Show that there exists a prime divisor $p$ of $M_\ell$ such that $\left(\frac{2}{p}\right) = -1$. 