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Elliptic curve primality tests for Fermat and related primes

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Abstract

We use elliptic curves with complex multiplication to develop primality tests for Fermat primes and for primes of the form $3^{2^\ell} - 3^{2^{\ell-1}} + 1$ and $2^{2^\ell} - 2^{2^{\ell-1}} + 1$.

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1. Introduction

Fermat numbers are integers of the form $F_\ell = 2^{2^\ell} + 1$. Prime numbers of this form are called Fermat primes. Although it is not known if there are infinitely many Fermat primes, primality of a given Fermat number can be verified by Pepin's test. It is an efficient and elegant test, based on the following coincidence: If F_ℓ is prime then the multiplicative group of the finite field \mathbb{F}_{F_ℓ} is a cyclic group of order 2^{2^ℓ} , a pure power of 2, and this group is always generated by 3.

The first goal of this paper is to develop a test for Fermat primes using the elliptic curve $y^2 = x^3 - x$. The test is based on the following: If F_ℓ is a prime then the group $E(F_\ell)$ of points modulo F_ℓ on the elliptic curve has (again) order 2^{2^ℓ} . This group is not cyclic. However, something just as interesting holds in this case. More precisely, if $p \equiv 1 \pmod{4}$ is a prime

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(a Fermat prime, for example) then $E(p)$ admits a complex multiplication by $\mathbb{Z}[i]$, the ring of Gaussian integers. We show that

$$E(F_\ell) \cong \mathbb{Z}[i]/(1+i)^{2^\ell}$$

as $\mathbb{Z}[i]$ -modules. Let E_n be the quadratic twist $ny^2 = x^3 - x$ of our curve E . If n is a square modulo F_ℓ then $E_n(F_\ell)$ is isomorphic to $E(F_\ell)$. One can now pick n and a rational point P of infinite order on E_n generating $E_n(F_\ell)$, as a $\mathbb{Z}[i]$ -module. For example, we can take $P = (5, 2)$ on the curve E_{30} . This data can be then used to formulate and prove a test for Fermat numbers similar to Pepin's test.

A version of the test can be perhaps best described in terms of Gaussian integers. Note that F_ℓ factors $F_\ell = f_\ell \cdot \bar{f}_\ell$ where $f_\ell = 2^{2^{\ell-1}} + i$. Starting with $x_1 = 5$ (the x -coordinate of the point P) define a sequence of Gaussian integers x_m modulo f_ℓ by a recursion formula (complex multiplication by $1 + i$)

$$x_{m+1} = \frac{1}{2} \left(\frac{x_m}{i} + \frac{i}{x_m} \right).$$

Then F_ℓ is prime if and only if x_m is relatively prime to f_ℓ for all $m = 1, \dots, 2^\ell - 1$ and

$$x_{2^\ell} \equiv 0 \pmod{f_\ell}.$$

It is interesting to note that this recursion does not depend on the choice of n . Moreover, there are other choices for the initial value x_1 for which the test works. A similar phenomenon occurs for the Lucas–Lehmer test for Mersenne numbers.

We then move to two other families of integers. We use elliptic curves of the form $y^2 = x^3 + \frac{D}{4}$ to develop two tests, for integers of the form

$$3^{2^\ell} - 3^{2^{\ell-1}} + 1 \quad \text{and} \quad 2^{2^\ell} - 2^{2^{\ell-1}} + 1,$$

respectively. Again, the main role is played by complex multiplication by $\mathbb{Z}[\omega]$, the ring of Eisenstein integers.

2. Pepin's test

In this section we quickly review Pepin's test for Fermat primes. The proof is presented in a way that generalizes to elliptic curves.

Proposition 1. *A Fermat number $F_\ell = 2^{2^\ell} + 1$ is prime if and only if*

$$3^{\frac{F_\ell-1}{2}} \equiv -1 \pmod{F_\ell}.$$

Proof. Assume first that the congruence holds. Let p be a prime dividing F_ℓ . Then

$$3^{\frac{F_\ell-1}{2}} \equiv -1 \pmod{p}$$

and, after squaring both sides of the congruence,

$$3^{F_\ell-1} \equiv 1 \pmod{p}.$$

In particular, the order of 3 modulo p divides $F_\ell - 1$. Since $F_\ell - 1 = 2^{2^\ell}$, a pure power of 2, any proper divisor of $F_\ell - 1$ is a divisor of $\frac{F_\ell-1}{2}$. Since $3^{\frac{F_\ell-1}{2}} \not\equiv 1 \pmod{p}$, the order of 3 modulo p is exactly $F_\ell - 1$. On the other hand, the order of 3 modulo p is less than or equal to $p - 1$. This implies

$$F_\ell - 1 \leq p - 1$$

or $F_\ell \leq p$. It follows that F_ℓ is prime.

The converse is more interesting, for it shows why the test really works. Assume that F_ℓ is prime. Note that $F_\ell \equiv 2 \pmod{3}$. By quadratic reciprocity, 3 is not a square modulo F_ℓ :

$$\left(\frac{3}{F_\ell}\right) = \left(\frac{F_\ell}{3}\right) = \left(\frac{2}{3}\right) = -1.$$

Note that the group $\mathbb{F}_{F_\ell}^\times$ has order equal to a pure power of 2 and is cyclic. Since 3 is not a square mod F_ℓ , it must be a generator of $\mathbb{F}_{F_\ell}^\times$. It follows that

$$3^{\frac{F_\ell-1}{2}}$$

is an element of order 2 in $\mathbb{F}_{F_\ell}^\times$. But -1 is the only element of order 2, and this completes the converse. The test is proved. \square

Of course, the number 3 can be replaced by any other number, provided it is not a square modulo F_ℓ . For example, if $F_\ell > 5$ then we can replace 3 by 5. Indeed, if $F_\ell > 5$ then $F_\ell \equiv 2 \pmod{5}$ and, by quadratic reciprocity,

$$\left(\frac{5}{F_\ell}\right) = -1.$$

3. The curve $y^2 = x^3 - x$

We denote the group of the elliptic curve $y^2 = x^3 - x$ over the finite field \mathbb{F}_p by $E(p)$. The discriminant of E is 2^6 . In particular E has a good reduction modulo any odd prime p .

Proposition 2. *Let p be an odd prime. If $p \equiv 1 \pmod{4}$ then $|E(p)| = p + 1 - 2a$ where $p = a^2 + b^2$ and $a + bi \equiv 1 \pmod{2 + 2i}$.*

Proof. See [I&R, p. 307]. \square

Corollary 3. *For $\ell > 1$ if $F_\ell = 2^{2^\ell} + 1$ is prime, then the group $E(F_\ell)$ satisfies $|E(F_\ell)| = 2^{2^\ell}$.*

Proof. Notice that $F_\ell = 1^2 + (2^{2^{\ell-1}})^2$ and that $\ell > 1 \Rightarrow 4 \mid 2^{2^{\ell-1}}$, which yields the congruence

$$1 + 2^{2^{\ell-1}}i \equiv 1 \pmod{2 + 2i}.$$

Thus by Proposition 2, $|E(F_\ell)| = F_\ell + 1 - 2 \cdot 1 = 2^{2^\ell}$. \square

In order to better understand the structure of the group $E(F_\ell)$, we now introduce a complex multiplication on our curve. If $p \equiv 1 \pmod{4}$ let $\pm i$ denote the primitive fourth roots of unity in the finite field \mathbb{F}_p . The action

$$i : (x, y) \rightarrow (-x, iy)$$

is an endomorphism and thus turns $E(p)$ into a $\mathbb{Z}[i]$ -module. Critical to us is the action of $1 + i$ on our curve. This is a degree 2 endomorphism of the elliptic curve. The only non-trivial point annihilated by $1 + i$ is

$$Q = (0, 0).$$

Proposition 4. Assume that $\ell > 1$ and F_ℓ is prime. Then

$$E(F_\ell) \cong \mathbb{Z}[i]/(1 + i)^{2^\ell}$$

as $\mathbb{Z}[i]$ -modules.

Proof. First $E(F_\ell)$ is a finitely generated $\mathbb{Z}[i]$ -module and so is isomorphic to the additive group:

$$\mathbb{Z}[i]/(\alpha_1) \oplus \mathbb{Z}[i]/(\alpha_2) \oplus \cdots \oplus \mathbb{Z}[i]/(\alpha_k)$$

for some $k \in \mathbb{N}$ and $\{\alpha_j\} \subseteq \mathbb{Z}[i]$. Now each $\mathbb{Z}[i]/(\alpha_j)$ is a subgroup of $E(F_\ell)$ hence $|\mathbb{Z}[i]/(\alpha_j)| = N(\alpha_j)$ divides the order of $E(F_\ell)$. By Corollary 3, $|E(F_\ell)| = 2^{2^\ell}$ so that $N(\alpha_j) = \alpha_j \bar{\alpha}_j$ must be a power of $2 = -i(1 + i)^2$. By uniqueness of factorization, for every j there exists in integer m_j such that

$$(\alpha_j) = ((1 + i)^{m_j}).$$

Finally, since multiplication by $1 + i$ is a degree 2 map, the annihilator of $1 + i$ in $E(F_\ell)$ has two elements. This implies that $k = 1$. The proposition is proved. \square

We now know that the $E(F_\ell)$ is a cyclic $\mathbb{Z}[i]$ -module. In order to build the test we need a point P that generates this module. This is accomplished as follows. Let E_n be the quadratic twist $ny^2 = x^3 - x$ of our elliptic curve E . Here n can be picked to be an integer or a rational number. If n is a non-zero square modulo F_ℓ then

$$(x, y) \mapsto (x, n^{\frac{1}{2}} \cdot y)$$

is an isomorphism of $\mathbb{Z}[i]$ -modules $E_n(F_\ell)$ and $E(F_\ell)$. Rational points on the curve E_n (for some n) are easy to construct. One picks a value for x and factors a square out of $x^3 - x$. For

example, if $x = 5$, then $5^3 - 5 = 120 = 30 \cdot 2^2$. This shows that $P = (5, 2)$ is a rational point on the curve E_{30} . Moreover,

$$\left(\frac{30}{F_\ell}\right) = \left(\frac{2}{F_\ell}\right) \cdot \left(\frac{3}{F_\ell}\right) \cdot \left(\frac{5}{F_\ell}\right) = 1 \cdot (-1) \cdot (-1) = 1$$

and 30 is a square modulo F_ℓ .

We need explicit formulae for the multiplication by $1 + i$ on the curve $ny^2 = x^3 - x$. The slope of the line through (x, y) and $(-x, iy)$ is

$$A = \frac{(1 - i)y}{2x}.$$

One now easily checks that $(1 + i) \cdot (x, y) = (x', y')$ where

$$\begin{cases} x' = nA^2 = \frac{1}{2} \left(\frac{x}{i} + \frac{i}{x} \right), \\ y' = -y - A(x' - x). \end{cases} \tag{1}$$

Proposition 5. *Let $\ell > 1$ be such that F_ℓ is prime. Then the rational point $P = (5, 2)$ is a generator of the $\mathbb{Z}[i]$ -module*

$$E_{30}(F_\ell) \cong \mathbb{Z}[i]/(1 + i)^{2^\ell}.$$

Proof. We must show that there is no point R in $E_{30}(F_\ell)$ such that

$$(5, 2) \equiv (1 + i) \cdot R \pmod{F_\ell}.$$

If there is such a point R , then the formula (1) for the action of $1 + i$ on the curve $30y^2 = x^3 - x$ implies that

$$5 \equiv 30 \cdot A^2 \pmod{F_\ell}.$$

Since 30 is a square modulo F_ℓ and 5 is not, this is a contradiction. The proposition is proved. \square

Of course, there are other choices for n and P . We can pick $x = 7$. Since $7^3 - 7 = 21 \cdot 4^2$, we have a rational point $P = (7, 4)$ on the curve E_{21} : $21y^2 = x^3 - x$. Using the quadratic reciprocity one easily shows that 7 is not a square modulo Fermat primes and $21 = 3 \cdot 7$ is a square modulo Fermat primes. It follows that the previous proposition holds with the rational point $P = (7, 4)$ on the curve E_{21} .

4. Elliptic curve test for Fermat primes

In this section we develop a test for Fermat primes using the curve $30y^2 = x^3 - x$ and the point $P = (5, 2)$. It is natural to state the test in terms of Gaussian integers. Note that we have a factorization in $\mathbb{Z}[i]$.

$$F_\ell = f_\ell \cdot \bar{f}_\ell$$

where $f_\ell = 2^{2^{\ell-1}} + i$. Now, F_ℓ is a prime integer if and only if f_ℓ is a Gaussian prime. Recall that $Q = (0, 0)$ is the unique point on the curve $30y^2 = x^3 - x$ of order $1 + i$.

Theorem. *Let $P = (5, 2)$ be a point on the curve $E_{30}: 30y^2 = x^3 - x$. Let $\ell > 1$. Then the Fermat number $F_\ell = 2^{2^\ell} + 1$ is prime if and only if*

$$(1 + i)^{2^\ell - 1} \cdot P \equiv Q \pmod{f_\ell}$$

where $f_\ell = 2^{2^{\ell-1}} + i$.

Proof. Assume that the congruence holds. We need to show that F_ℓ is prime. If not, then there exists a prime factor p of F_ℓ such that $p < \sqrt{F_\ell}$. The prime p is clearly not equal to 3 or 5. In particular the curve has a good reduction modulo p .

Since p divides F_ℓ ,

$$(2^{2^{\ell-1}})^2 \equiv -1 \pmod{p}$$

which shows that -1 is a square mod p . In particular, p is not a Gaussian prime and we can write $p = \pi \bar{\pi}$. Without loss of generality, assume that π divides f_ℓ . By assumption, we also have the congruence

$$(1 + i)^{2^\ell - 1} \cdot P \equiv Q \pmod{\pi}.$$

Multiplying both sides of this congruence by $1 + i$ gives

$$(1 + i)^{2^\ell} \cdot P \equiv O \pmod{\pi}$$

where O is the identity element in $E(\pi)$. It follows that P generates a $\mathbb{Z}[i]$ -submodule of $E_{30}(\pi)$ isomorphic to $\mathbb{Z}[i]/((1 + i)^{2^\ell})$. The order of this module is $N((1 + i)^{2^\ell}) = 2^{2^\ell} = F_\ell - 1$ which implies $F_\ell - 1 \leq |E_{30}(\pi)|$. However, by Hasse's estimate, the order of $E(\pi)$ is bounded by

$$|E_{30}(\pi)| \leq p + 1 + 2\sqrt{p} = (\sqrt{p} + 1)^2.$$

Keeping in mind $p^2 < F_\ell$ we have created the scenario $p^2 - 1 < (\sqrt{p} + 1)^2$ which does not hold for any prime $p > 2$. F_ℓ is odd thus F_ℓ must have been prime to begin with.

For the reverse direction, assume F_ℓ is a prime. Notice that $E_{30}(F_\ell)$ is isomorphic to $E_{30}(f_\ell)$ via the natural isomorphism of the finite fields $\mathbb{Z}/(F_\ell)$ and $\mathbb{Z}[i]/(f_\ell)$. By Proposition 4,

$$E_{30}(f_\ell) \cong \mathbb{Z}[i]/((1 + i)^{2^\ell}),$$

and by Proposition 5, the point P generates this $\mathbb{Z}[i]$ -module. It follows that $(1 + i)^{2^\ell - 1} \cdot P$ is an element of order $1 + i$. Since $Q = (0, 0)$ is the only such element, we are done. \square

The test can be further rewritten as follows. Let

$$(x_m, y_m) = (1 + i)^{m-1} \cdot P.$$

By our formula for the multiplication by $1 + i$, the numbers x_m are given by a quadratic recursion

$$x_{m+1} = \frac{1}{2} \left(\frac{x_m}{i} + \frac{i}{x_m} \right)$$

starting with $x_1 = 5$. In this way we arrive to a version of the test alluded to in the introduction: The Fermat number F_ℓ is prime if and only if x_m is relatively prime to f_ℓ for all $m = 1, \dots, 2^\ell - 1$ and x_{2^ℓ} is 0 modulo f_ℓ .

As an example we calculate the first few terms in this sequence and use them to test the primality of some small Fermat numbers. The sequence starts,

$$\{x_i\} = \left\{ 5, -\frac{12i}{5}, -\frac{169}{120}, \frac{14161i}{40560}, \dots \right\}.$$

Modulo $f_2 = 4 + i$ this sequence becomes $\{5, 4, 1, 0\}$, thus $F_2 = 17$ is prime. Modulo $f_5 = 2^{16} + i$ we calculate $x_{32} \equiv 3436246100 \not\equiv 0 \pmod{f_5}$ thus F_5 is not prime as Euler first calculated in 1732. Of course our test does not necessarily factor the Fermat numbers as Euler did.

There are other choices for n and the starting point P of course. One can take $P = (\frac{3}{2}, \frac{1}{2})$, and $n = \frac{15}{2}$ giving $x_1 = \frac{3}{2}$ as the starting point. Because the formula is independent of n , one uses the same recurrence rule as before! It is of no computational benefit however to change the curve and starting point.

5. The curve $y^2 = x^3 + \frac{D}{4}$

Let C be the elliptic curve $y^2 = x^3 + \frac{D}{4}$. The discriminant of C is $-3^3 D^2$. In particular C has a good reduction modulo any odd prime p not dividing $3D$. We let $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$ represent the third root of unity in \mathbb{C} , as usual we have $\omega^2 = \bar{\omega}$. The ring $\mathbb{Z}[\omega]$ is known as the ring of Eisenstein integers which consists of $\{a + b\omega \mid a, b \in \mathbb{Z}\}$. The norm of this ring is given by $N(a + b\omega) = (a + b\omega) \cdot \overline{(a + b\omega)} = a^2 - ab + b^2$.

If $p \equiv 1 \pmod{3}$ then, abusing notation, let ω denote a primitive third root of unity in the finite field \mathbb{F}_p . The action

$$\omega : (x, y) \rightarrow (\omega x, y)$$

is a degree three endomorphism of $C(p)$, and thus turns $C(p)$ into a $\mathbb{Z}[\omega]$ -module.

Proposition 6. *Let p be an odd prime not dividing $3D$. If $p \equiv 1 \pmod{3}$, let $p = \pi \bar{\pi}$ with $\pi \in \mathbb{Z}[\omega]$, and $\pi \equiv \bar{\pi} \equiv 2 \pmod{3}$. Then,*

$$|C(p)| = p + 1 + \left(\frac{\bar{D}}{\pi}\right)_6 \pi + \left(\frac{D}{\pi}\right)_6 \bar{\pi}.$$

Proof. See [I&R, p. 305]. \square

We now turn our attention to two more sets of integers. Define

$$\begin{cases} K_\ell = 3^{2^\ell} - 3^{2^{\ell-1}} + 1, \\ J_\ell = 2^{2^\ell} - 2^{2^{\ell-1}} + 1. \end{cases}$$

We have $K_\ell \equiv J_\ell \equiv 1 \pmod{3}$. Setting

$$\begin{cases} k_\ell = -1 - 3^{2^{\ell-1}} \omega, \\ j_\ell = \omega + 2^{2^{\ell-1}} \bar{\omega} \end{cases}$$

we have factorizations $J_\ell = j_\ell \cdot \bar{j}_\ell$ and $K_\ell = k_\ell \cdot \bar{k}_\ell$. Note that $k_\ell \equiv j_\ell \equiv 2 \pmod{3}$.

Corollary 7. *Let $\ell > 1$.*

- (i) *Let $D = 1$. If K_ℓ is prime then $|C(K_\ell)| = 3^{2^\ell}$.*
- (ii) *Let $D = 4n^3$. If J_ℓ is prime and $(\frac{n}{J_\ell})_2 = -1$ then $|C(J_\ell)| = 2^{2^\ell}$.*

Proof. For (i) if K_ℓ is prime then k_ℓ is an Eisenstein prime. By Proposition 6 we have, with $\pi = k_\ell$,

$$|C(K_\ell)| = K_\ell + 1 + k_\ell + \bar{k}_\ell = 3^{2^\ell}. \tag{2}$$

For (ii), if J_ℓ is prime then j_ℓ is an Eisenstein prime. Recall that 2 is a prime in $\mathbb{Z}[\omega]$. By cubic reciprocity (see [I&R]),

$$\left(\frac{2}{j_\ell}\right)_3 = \left(\frac{\omega + 2^{2^{\ell-1}} \bar{\omega}}{2}\right)_3 = \left(\frac{\omega}{2}\right)_3 = \omega.$$

Next, note that the inclusion of \mathbb{Z} into $\mathbb{Z}[\omega]$ gives rise to an isomorphism of finite fields $\mathbb{Z}/(J_\ell)$ and $\mathbb{Z}[\omega]/(j_\ell)$. In particular, an integer n is a square modulo J_ℓ if and only if it is a square modulo j_ℓ . Therefore, since $D = 4n^3$ and n is not a square modulo J_ℓ , it follows that

$$\left(\frac{D}{j_\ell}\right)_6 = \left(\frac{2}{j_\ell}\right)_3 \left(\frac{n}{j_\ell}\right)_2 = -\omega.$$

Then by Proposition 6 we have,

$$|C(J_\ell)| = J_\ell + 1 + (-\bar{\omega})j_\ell + (-\omega)\bar{j}_\ell = 2^{2^\ell}. \tag{3}$$

This proves the corollary. \square

We now show that $n = 7$ satisfies the second condition of the corollary. By quadratic reciprocity we have

$$\left(\frac{7}{J_\ell}\right) = \left(\frac{J_\ell}{7}\right) (-1)^{\frac{7-1}{2} \frac{J_\ell-1}{2}} = \left(\frac{J_\ell}{7}\right).$$

Modulo 7, 2^{2^ℓ} alternates between the values 2 and 4. Thus either $J_\ell \equiv 2 - 4 + 1 \equiv -1 \pmod{7}$, or $J_\ell \equiv 4 - 2 + 1 \equiv 3 \pmod{7}$. In both cases we have $(\frac{J_\ell}{7}) = -1$ thus $(\frac{7}{J_\ell}) = -1$. We get, by part (ii) of Corollary 7, that $|C(J_\ell)| = 2^{2^\ell}$ for $D = 7^3$.

6. The structure of the group $C(K_\ell)$

Recall that C is the elliptic curve $y^2 = x^3 + \frac{D}{4}$. Assume that K_ℓ is a prime number. Since $K_\ell \equiv 1 \pmod{3}$ the group $C(K_\ell)$ is a $\mathbb{Z}[\omega]$ -module.

Proposition 8. *Let C be the elliptic curve $y^2 = x^3 + \frac{1}{4}$. If K_ℓ is prime then*

$$C(K_\ell) \cong \mathbb{Z}[\omega]/(\omega - \omega^2)^{2^{\ell-1}}$$

as $\mathbb{Z}[\omega]$ -modules.

Proof. This is proved in the same way as Proposition 4. We observe that $|C(K_\ell)| = 3^{2^\ell}$ and that $(\omega - \omega^2)$ is the unique prime ideal in $\mathbb{Z}[\omega]$ with norm equal to a power of 3 and multiplication by $\omega - \omega^2$ is a degree 3 endomorphism. \square

Next, we want to find a generator of the $\mathbb{Z}[\omega]$ -module $C(K_\ell)$. To this end, let C_n be the cubic twist $y^2 = nx^3 + \frac{1}{4}$. Clearly, if n is a cube, then

$$(x, y) \mapsto (n^{\frac{1}{3}} \cdot x, y)$$

gives an isomorphism between $\mathbb{Z}[\omega]$ -modules $C_n(K_\ell)$ and $C(K_\ell)$. We can construct rational points on C_n by picking y and then factoring a cube out of $y^2 - \frac{1}{4}$. Trying $y = 2$ we get $x = \frac{1}{2}$ and $n = 30$. We have constructed the point $P = (\frac{1}{2}, 2)$ on the curve

$$y^2 = 30x^3 + \frac{1}{4}.$$

Of course, we must show that 30 is a cube modulo K_ℓ or, equivalently, modulo k_ℓ . Recall that 2 is a prime in $\mathbb{Z}[\omega]$. By cubic reciprocity,

$$\left(\frac{2}{k_\ell}\right)_3 = \left(\frac{-1 - 3^{2^{\ell-1}}\omega}{2}\right)_3 = \left(\frac{-1 - \omega}{2}\right)_3 = \omega^2.$$

Next, 5 is also a prime in $\mathbb{Z}[\omega]$, and $5 \equiv 2 \pmod{3}$. Then by cubic reciprocity

$$\left(\frac{5}{k_\ell}\right)_3 = \left(\frac{-1 - 3^{2^{\ell-1}}\omega}{5}\right)_3 = \left(\frac{-1 - \omega}{5}\right)_3 = (\omega^2)^{\frac{5^2-1}{3}} = \omega.$$

Note that, since $K_\ell \equiv 1 \pmod{9}$ for $\ell > 1$, ω is a cube modulo K_ℓ . Since $1 - \omega$ is a cube modulo k_ℓ by p. 114 in [I&R], we conclude that $-3 = (\omega - \omega^2)^2$ and 3 are cubes modulo k_ℓ :

$$\left(\frac{3}{k_\ell}\right)_3 = 1.$$

Putting these calculations together yields,

$$\left(\frac{30}{k_\ell}\right)_3 = \left(\frac{2}{k_\ell}\right)_3 \cdot \left(\frac{5}{k_\ell}\right)_3 \cdot \left(\frac{3}{k_\ell}\right)_3 = 1. \tag{4}$$

This shows that 30 is a cube modulo any prime K_ℓ .

Using the addition formula for our curve, $y^2 = nx^3 + \frac{1}{4}$ we see $(\omega - \omega^2) \cdot (x, y) = (x', y')$ where

$$\begin{cases} x' = \frac{1}{n}A^2 + x = -\frac{nx^3 + 1}{3nx^2}, \\ y' = -y - A(x' - \omega x) = y(\omega - \omega^2) + \frac{8}{3(\omega - \omega^2)} \cdot \frac{y^3}{y^2 - \frac{1}{4}} \end{cases} \quad (5)$$

and A is the slope of the line through the points $(\omega x, y)$ and $(\omega^2 x, -y)$, given by

$$A = \frac{2y}{(\omega - \omega^2)x}.$$

Proposition 9. *Let $\ell > 1$ be such that K_ℓ is prime. Then the rational point $P = (\frac{1}{2}, 2)$ is a generator of the $\mathbb{Z}[\omega]$ -module*

$$C_{30}(K_\ell) \cong \mathbb{Z}[\omega]/(\omega - \omega^2)^{2^\ell}.$$

Proof. We must show that there is no point R in $C_{30}(K_\ell)$ such that

$$\left(\frac{1}{2}, 2\right) \equiv (\omega - \omega^2) \cdot R \pmod{K_\ell}.$$

Assume there was such a point, $R = (x, y)$. By the formula (5) for multiplication by $\omega - \omega^2$, solving for $R = (x, y)$ amounts to solving a cubic equation

$$30x^3 + 45x^2 + 1 = 0.$$

We substitute $z = \frac{1}{x}$ to get the equation,

$$z^3 + 45z + 30 = 0. \quad (6)$$

Assume that R in $C_{30}(K_\ell)$ is one solution. If $S \in C_{30}(K_\ell)$ satisfies $(\omega - \omega^2) \cdot S \equiv O \pmod{K_\ell}$, then $R + S$ is another solution. There are three such S , thus we get three distinct solutions to $(\omega - \omega^2) \cdot R \equiv P \pmod{K_\ell}$. It follows that the polynomial (6) splits in \mathbb{F}_{K_ℓ} .

On the other hand, one can solve (6) as in [D&F]. We see $p = 45$ and $q = 30$ so that the discriminant $D = -4p^3 - 27q^2 = -3(2^3 \cdot 3^2 \cdot 5)^2$. Now this equation has a solution if and only if the value,

$$\frac{-27}{2}q + \frac{3}{2}\sqrt{-3D} = 3^5 \cdot 5$$

is a cube in \mathbb{F}_{K_ℓ} . Using our previous computations we find,

$$\left(\frac{3^5 \cdot 5}{k_\ell}\right)_3 = \left(\frac{5}{k_\ell}\right)_3 = \omega.$$

This shows that (6) does not have a root in \mathbb{F}_{K_ℓ} . It follows that the polynomial in (6) does not actually split over \mathbb{F}_{K_ℓ} , and so there is no solution to $(\omega - \omega^2) \cdot R \equiv P \pmod{K_\ell}$. This finishes the proof that P generates $C_{30}(K_\ell)$. \square

7. Elliptic curve test for the primes K_ℓ

We now develop a test for the numbers K_ℓ using the curve $y^2 = 30x^3 + \frac{1}{4}$ and the point $P = (\frac{1}{2}, 2)$. It is more natural to state the test is in terms of Eisenstein integers. Recall that we have a factorization in $\mathbb{Z}[\omega]$.

$$K_\ell = k_\ell \cdot \bar{k}_\ell$$

where $k_\ell = -1 - 3^{2^{\ell-1}}\omega$. Now, K_ℓ is a prime integer if and only if k_ℓ is an Eisenstein prime. Note that the points of order $\omega - \omega^2$ on $y^2 = 30x^3 + \frac{1}{4}$ are

$$\left(0, \pm \frac{1}{2}\right).$$

Theorem. *Let $P = (\frac{1}{2}, 2)$ be a point on the elliptic curve $C_{30}: y^2 = 30x^3 + \frac{1}{4}$. Let $\ell > 1$. The number $K_\ell = 3^{2^\ell} - 3^{2^{\ell-1}} + 1$ is prime if and only if*

$$(\omega - \omega^2)^{2^{\ell-1}} \cdot P \equiv \left(0, \pm \frac{1}{2}\right) \pmod{k_\ell}.$$

Proof. Assume the congruence holds. Suppose K_ℓ is not prime. Then there exists a prime factor p of K_ℓ such that $p < \sqrt{K_\ell}$. The prime cannot be 2, 3 or 5 so the curve has good reduction modulo p . Since p divides K_ℓ ,

$$0 \equiv 3^{2^\ell} - 3^{2^{\ell-1}} + 1 \equiv x^2 - x + 1 \pmod{p}$$

which shows there is a non-trivial cube root of one mod p . This shows $p \equiv 1 \pmod{3}$. Then $p = \pi \bar{\pi}$ for some Eisenstein prime π . We can assume that π divides k_ℓ . We get the congruence

$$(\omega - \omega^2)^{2^{\ell-1}} \cdot P \equiv \left(0, \pm \frac{1}{2}\right) \pmod{\pi}.$$

Multiplying both sides of the congruence by $\omega - \omega^2$ yields

$$(\omega - \omega^2)^{2^\ell} \cdot P \equiv O \pmod{\pi}.$$

Thus P generates a $\mathbb{Z}[\omega]$ -submodule of $C_{30}(\pi)$ isomorphic to $\mathbb{Z}[\omega]/((\omega - \omega^2)^{2^\ell})$. The order of this module is $N((\omega - \omega^2)^{2^\ell}) = 3^{2^\ell}$, so we must have

$$|C_{30}(\pi)| \geq 3^{2^\ell}.$$

By Hasse’s estimate we also have $|C_{30}(\pi)| \leq (\sqrt{p} + 1)^2$. Combining these with the earlier remark, $p \leq \sqrt{K_\ell} < 3^{2^{\ell-1}}$, we get the inequality,

$$3^{2^\ell} \leq |C_{30}(p)| \leq (\sqrt{p} + 1)^2 < 3^{2^{\ell-1}} + 2 \cdot 3^{2^{\ell-2}} + 1$$

which holds for no $\ell > 1$. This is a contradiction, so it must be that K_ℓ is prime.

For the other direction assume that K_ℓ is prime. By Proposition 8,

$$C_{30}(k_\ell) \cong C_{30}(K_\ell) \cong \mathbb{Z}[\omega]/((\omega - \omega^2)^{2^\ell}),$$

and by Proposition 9 the point P generates this $\mathbb{Z}[\omega]$ -module. Therefore $(\omega - \omega^2)^{2^\ell - 1} \cdot P$ is an element of order $\omega - \omega^2$ and this is $(0, \frac{1}{2})$ or $(0, -\frac{1}{2})$. The theorem follows. \square

We now formulate the more elegant version of this test. Define $y_1 = 2$, and recursively define

$$y_{m+1} = y_m(\omega - \omega^2) + \frac{8}{3(\omega - \omega^2)} \cdot \frac{y_m^3}{y_m^2 - \frac{1}{4}}.$$

Then the number $K_\ell = 3^{2^\ell} - 3^{2^{\ell-1}} + 1$ is prime if and only if $y_m^2 - \frac{1}{4}$ is relatively prime to K_ℓ for $m = 1, \dots, 2^\ell - 1$ and $y_{2^\ell}^2 - \frac{1}{4}$ is zero mod K_ℓ .

8. The structure of the group $C(J_\ell)$

The curve $y^2 = x^3 + n^3$ ($D = 4n^3$) can be rewritten as $ny^2 = x^3 + 1$ using the substitution

$$(x, y) \mapsto (n^{-1}x, n^{-2}y).$$

Let C_n denote this curve. Assume that J_ℓ is a prime number. Since $J_\ell \equiv 1 \pmod{3}$, the group $C_n(J_\ell)$ is a $\mathbb{Z}[\omega]$ -module.

Proposition 10. *If J_ℓ is prime and $(\frac{n}{J_\ell})_2 = -1$ then*

$$C_n(J_\ell) \cong \mathbb{Z}[\omega]/(2^{2^{\ell-1}})$$

as $\mathbb{Z}[\omega]$ -modules.

Proof. This is proved in the same way as Proposition 4, using that $|C_n(J_\ell)| = 2^{2^\ell}$ and observing that (2) is the unique prime ideal in $\mathbb{Z}[\omega]$ with norm equal to a power of 2 and multiplication by 2 is a degree 4 endomorphism. \square

For the primes J_ℓ we will need to make use of the duplication formula on our curve. The slope of the tangent line to (x, y) on the curve $ny^2 = x^3 + 1$ is defined to be

$$A = \frac{dy}{dx} = \frac{3x^2}{2ny}.$$

One now checks that the duplication formula takes the shape $2 \cdot (x, y) = (x', y')$ where

$$\begin{cases} x' = nA^2 - 2x = \frac{x^4 - 8x}{4(x^3 + 1)}, \\ y' = -y - A(x' - x). \end{cases} \quad (7)$$

We know that $n = 7$ is not a square modulo all primes J_ℓ . In particular, Proposition 10 holds for C_7 . This curve has a rational point $P = (3, 2)$.

Proposition 11. *Let $\ell > 1$ be such that J_ℓ is prime. Then the rational point $P = (3, 2)$ is a generator of the $\mathbb{Z}[\omega]$ -module*

$$C_7(J_\ell) \cong \mathbb{Z}[\omega]/(2^{2^{\ell-1}}).$$

Proof. It suffices to show that the equation

$$P = (3, 2) \equiv 2 \cdot R \pmod{J_\ell}$$

has no solution in $C_7(J_\ell)$. By the duplication formula (7), solving for $R = (x, y)$ amounts to solving a quartic equation

$$x^4 - 12x^3 - 8x - 12 = 0. \quad (8)$$

Assume that R in $C(J_\ell)$ is one solution. Now if $S \in C(J_\ell)$ satisfies $2 \cdot S \equiv O \pmod{J_\ell}$, then we have $R + S$ is another solution. There are four such S , thus we get four distinct solutions to $2 \cdot R \equiv P \pmod{J_\ell}$ in $C_7(J_\ell)$. It follows that the polynomial in (8) splits in \mathbb{F}_{J_ℓ} .

On the other hand, recall that $(\frac{7}{J_\ell}) = -1$, thus $\mathbb{F}_{J_\ell}[\sqrt{7}]$ is a degree 2 extension. In this extension we have the (tricky) factorization,

$$x^4 - 12x^3 - 8x - 12 = (x^2 - 2(3 + \sqrt{7})x - 2(2 + \sqrt{7})) \cdot (x^2 - 2(3 - \sqrt{7})x - 2(2 - \sqrt{7})).$$

If x_1, x_2 were the two roots of the first term in the right-hand side of this equation then we would have $x_1 + x_2 = 2(3 + \sqrt{7})$, and as such we see one of x_1, x_2 is not an element of \mathbb{F}_{J_ℓ} . It follows that the polynomial in (8) does not actually split over \mathbb{F}_{J_ℓ} , and so there is no solution to $2 \cdot R \equiv P \pmod{J_\ell}$. This finishes the proof that P generates $C_7(J_\ell)$. \square

9. Elliptic curve test for the primes J_ℓ

Finally, we develop a test for the numbers J_ℓ using the curve $7y^2 = x^3 + 1$ and the point $P = (3, 2)$. Again, it is more natural to state the test is in terms of Eisenstein integers. Recall that we have a factorization in $\mathbb{Z}[\omega]$.

$$J_\ell = j_\ell \cdot \bar{j}_\ell$$

where $j_\ell = \omega + 2^{2^{\ell-1}}\bar{\omega}$. Now, J_ℓ is a prime integer if and only if j_ℓ is an Eisenstein prime. Note that the points of order 2 on $7y^2 = x^3 + 1$ are of the form

$$(-\omega^j, 0).$$

Theorem. Let $P = (3, 2)$ be a point on the elliptic curve $C_7: 7y^2 = x^3 + 1$. Let $\ell > 1$. The number $J_\ell = 2^{2^\ell} - 2^{2^{\ell-1}} + 1$ is prime if and only if

$$2^{2^{\ell-1}-1} \cdot P \equiv (-\omega^j, 0) \pmod{j_\ell}.$$

Proof. Assume the congruence is true. Suppose J_ℓ is not prime. Then there exists a prime factor p of J_ℓ such that $p < \sqrt{J_\ell}$. The prime cannot be 3, nor 7 so the curve has good reduction modulo p . Since p divides J_ℓ ,

$$0 \equiv 2^{2^\ell} - 2^{2^{\ell-1}} + 1 \equiv x^2 - x + 1 \pmod{p}$$

which shows there is a non-trivial cube root of one mod p . This shows $p \equiv 1 \pmod{3}$ and we can write $p = \pi \bar{\pi}$ for an Eisenstein prime π . Without any loss of generality, we can assume that π divides j_ℓ . By assumption, we also get the congruence

$$2^{2^{\ell-1}-1} \cdot P \equiv (-\omega^j, 0) \pmod{\pi}.$$

Multiplying both sides of the congruence by 2 yields

$$2^{2^{\ell-1}} \cdot P \equiv O \pmod{p}.$$

Thus P generates a $\mathbb{Z}[\omega]$ -submodule of $C_7(\pi)$ isomorphic to $\mathbb{Z}[\omega]/(2^{2^{\ell-1}})$. The order of this module is $N(2^{2^{\ell-1}}) = 2^{2^\ell}$, so we must have

$$|C_7(\pi)| \geq 2^{2^\ell}.$$

By Hasse's estimate we also have $|C_7(\pi)| \leq (\sqrt{p} + 1)^2$. Combining these with the earlier remark, $p \leq \sqrt{J_\ell} < 2^{2^{\ell-1}}$, we get the awkward inequality,

$$2^{2^\ell} \leq |C_7(\pi)| \leq (\sqrt{p} + 1)^2 < 2^{2^{\ell-1}} + 2 \cdot 2^{2^{\ell-2}} + 1$$

which holds for no $\ell > 1$. This is a contradiction, so it must be that J_ℓ is prime.

For the other direction assume that J_ℓ is prime. By Proposition 10,

$$C_7(j_\ell) \cong C_7(J_\ell) \cong \mathbb{Z}[\omega]/(2^{2^{\ell-1}}),$$

and by Proposition 11 the point P generates this $\mathbb{Z}[\omega]$ -module. It follows that $2^{2^{\ell-1}-1} \cdot P$ is an element of order 2, hence of the form $(-\omega^j, 0)$. The theorem follows. \square

We can reformulate this test in a similar way to the first test *without* even using the Eisenstein integers. Define $x_1 = 3$, and then recursively define

$$x_{m+1} = \frac{x_m^4 - 8x_m}{4(x_m^3 + 1)}.$$

Then the number $J_\ell = 2^{2^\ell} - 2^{2^{\ell-1}} + 1$ is prime if and only if $x_m^3 + 1$ is relatively prime to J_ℓ for $n = 1, \dots, 2^{\ell-1} - 1$ and $x_{2^{\ell-1}}^3 + 1$ is zero mod J_ℓ .

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Further reading

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