1) The number 3-7643-9326-8 is obtained by transposing two consecutive digits of a valid ISBN number. Find that ISBN number.

Solution:

\[ \sum_{i=1}^{10} ix_i \equiv -6 \equiv 5 \pmod{11}. \]

Thus, we need to either increase the sum by 6 or decrease the sum by 5. The first could be accomplished by replacing 93 by 39 in the number above.

2) Use the Euclidean Algorithm to compute the multiplicative inverse of 56 modulo 131. Then solve

\[ 56x \equiv 29 \pmod{131}. \]

Solution:

\[ 131 = 2 \cdot 56 + 19 \]
\[ 56 = 3 \cdot 19 - 1 \]

Substituting \( 19 = 131 - 2 \cdot 56 \) into the second equation gives \( 6 \cdot 131 - 7 \cdot 56 = 1 \) which means that \(-7\) is the inverse of 56. Hence

\[ x \equiv (-7) \cdot 29 \equiv -72 \equiv 49 \pmod{131}. \]

3) Let \( a \) and \( b \) be two positive integers and \( LCM(a,b) \) the lowest common multiple of \( a \) and \( b \). Let \( m \) be a common multiple of \( a \) and \( b \). Show that \( LCM(a,b) \) divides \( m \).

Solution: Write

\[ m = qLCM(a,b) + r \]

where \( 0 \leq r < LCM(a,b) \). Since \( a \) divides \( m \) and \( LCM(a,b) \), it divides \( r = m - qLCM(a,b) \). Since \( b \) divides \( m \) and \( LCM(a,b) \), it divides \( r = m - qLCM(a,b) \). Hence \( r \) is a common multiple of \( a \) and \( b \). Thus \( r = 0 \).

4) The number 2 is a primitive root modulo 19. Compute the powers \( 2^I \) for \( I = 1, 2, \ldots, 18 \) to obtain the table for the discrete logarithm with base 2 for integers modulo 19. Then use the table to solve the equation

\[ x^5 \equiv 7 \pmod{19}. \]
Solution: The table gives that $\log_2(7) = 6$. The equation $x^5 \equiv 7 \pmod{19}$ implies

$$5\log_2(x) \equiv \log_2(7) \equiv 6 \pmod{18}.$$ 

One calculates easily that the inverse of 5 modulo 18 is 11. Thus

$$\log_2(x) \equiv 11 \cdot 6 \equiv 12 \pmod{18}.$$ 

The table now gives that $x = 11$.

5) Does the equation $x^2 - 4x - 27 = 0$ have a solution modulo 131? Hint: complete to a square, then use the quadratic reciprocity.

Solution: Completing to a square gives $(x - 2)^2 = 31$, so the question is whether 31 is a square modulo 131.

$$\left(\frac{31}{131}\right) = -\left(\frac{131}{31}\right) = -\left(\frac{7}{31}\right) = \left(\frac{31}{7}\right) = -\left(\frac{3}{7}\right) = \left(\frac{7}{3}\right) = 1$$

So 31 is indeed a square modulo 131.